AERONAUTICS

IN THEORY AND EXPERIMENT

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SECOND EDITION

NEW YORK
LONGMANS, GREEN & CO.
LONDON: EDWARD ARNOLD
1920

PREFACE TO THE FIRST EDITION

As a result of the stimulus of war, aeronautics has made such gigantic strides during the past few years that it has definitely attained the rank of a science, and will, it is anticipated, shortly take its proper position in the curriculum of every university and technical college.

The present volume is the outcome of an attempt to provide a sound and scientific treatment of the fundamental principles upon which the subject is based, and to indicate the lines of further development. The book, it is hoped, will thus satisfy a rapidly increasing demand. War conditions have unfortunately limited the authors considerably in the material of which they might have availed themselves, but they have striven throughout to present a sound theory, and only to utilise data as a means of exemplifying the application of the principles enunciated; it should accordingly be understood that certain of the data may require to be slightly modified when the present restrictions are removed. From this point of view, no apology is made for presenting in Plates I, II, III a machine, not quite of the latest type, for the purpose of explaining the relative importance and positions of the various parts, and the form of structure of wing and fuselage. Historically it illustrates the stage to which the science had developed just prior to the outbreak of hostilities.

The importance of a thorough conception of the nature of fluid motion in the study of aeronautics cannot be over-emphasised, and the authors hope that the manner of treatment adopted will commend itself to the reader. Unfortunately the mathematical development of this branch of the subject leads to analysis of considerable complexity and consequently the motion of a perfect fluid only has been dealt with to any extent in this manner, although a few of the simpler cases of the flow of a viscous fluid have also been included. For the student of moderate mathematical ability the treatment of the subject in Chapter II and the latter part of Chapter III, where the fluid is regarded from the molecular point of view, should suffice. Associated with this, is the subject matter of Chapter IV, where the theoretical basis of the transition from the results of model experimental work to those for the full scale is developed and

exemplified by illustrations from fluid motion, etc. The actual application of the theory to the aerofoil and to the propeller is dealt with in detail in the chapters treating specifically with these parts.

The subject matter of Chapter VIII on Strength of Construction it is felt is considerably curtailed, as it must be at the present time, but it is proposed in a future edition to amplify this section by the inclusion of later work on the critical loading of struts and aeroplane structures generally, the calculation of the force distributions on fuselages, and much experimental research that has lately been brought to fruition.

No text book on Aeronautics can be complete without a thorough discussion of stability both from the theoretical and the experimental standpoint, so ably investigated by Bryan and Bairstow. Once again the mathematics may appear formidable, but as far as possible the task of the student has been lightened by full explanations of the mathematical methods adopted. To this end, the equations of moving axes have been derived as simply and as clearly as possible in order that the student whose knowledge of mathematics is limited may succeed in driving through this section of the subject without the heart-breaking despair frequently associated with it. Many of the equations could have been derived equally well with reference to fixed axes, but the work in its later development, e.g. in the treatment of laterals, would have become exceedingly complicated.

Every serious student must realise that a mastery of stability is an essential for the study of the problem of safe flight, and perseverance is the only advice that can be offered.

The authors would like to express their thanks to Sir Richard Glazebrook, F.R.S., Director of the National Physical Laboratory, to Dr Stanton, F.R.S., Superintendent of the Aerodynamics Division of the N.P.L., and to Mr Bairstow, F.R.S., C.B.E., for their valuable criticism of the manner of presentation, and for the facilities afforded in the use of material. Thanks are also due to the Advisory Committee for Aeronautics for the use of the half-tones and for permission to publish much of the subject matter.

TEDDINGTON,
February, 1918.

PREFACE TO THE SECOND EDITION

THE two years that have elapsed since the issue of the first edition of this book have seen a flood of aeronautical literature poured upon the market, covering almost every aspect of the problem of flight both in relation to navigation and design. The authors have consequently not attempted in this edition to provide anything of the nature of an elaborate compendium of up-to-date design data, but have rather scrutinised carefully the fundamental principles enunciated in the previous edition where fuller knowledge and experience have provided a wider and deeper insight. In all instances more modern and more accurate experimental data have been substituted in the illustrative examples, where necessary, but the general sequence of the book is substantially unaltered. Criticism and review have confirmed the authors in the opinion that the method of development of the subject adopted in the first edition is both the most logical and the most instructive, and they have seen no reason for modification in this respect. At the same time certain sections have been considerably amplified and extended. Part III, dealing with the strength of aeroplane structures, has been completely rewritten and recast on the basis of work on this subject carried out by the authors themselves, and covers the problems of stress distribution in wings and fuselage structures, the critical loading of struts and the laws of similarity in aeroplane structures. Although some of this may be well without the scope of the aeroplane designer, a knowledge of these principles and methods is an essential to the serious student, and for him the present work is intended to serve as a basis.

The authors note with pleasure the rapid fulfilment of their anticipation, voiced in the earlier edition, that shortly the science of aeronautics would take its proper position in the curriculum of university and technical colleges.

TEDDINGTON,
July, 1920.

CONTENTS

CHAPTER I

INTRODUCTION

PART I

CHAPTER II

FLUID MOTION; ELEMENTARY THEORY AND EXPERIMENT

CHAPTER III

MATHEMATICAL THEORY OF FLUID MOTION

CHAPTER IV

FROM MODEL TO FULL SCALE

PART II

CHAPTER V

THE AEROFOIL

Characteristics required for special types of machines

CHAPTER VI

THE AEROFOIL (CONTD.)

Experimental Investigations

CHAPTER VII

STRUCTURAL PARTS AND CONTROLS

PART III

CHAPTER VIII

STRENGTH OF CONSTRUCTION

Analysis of Stress Distribution in Aeroplane Members

CHAPTER IX

CRITICAL BEHAVIOUR OF STRUCTURES

PART IV

CHAPTER X

THE AIRSCREW

PART V

CHAPTER XI

STEADY MOTION IN FLIGHT

CHAPTER XII

STABILITY—MATHEMATICAL THEORY

CHAPTER XIII

STABILITY—EXPERIMENTAL

CHAPTER XIV

THE COMPLETE MACHINE

Conclusion

LIST OF PLATES

FOLDING PLATES AT THE END OF THE BOOK

CHAPTER I

INTRODUCTION

§ **1.** The development of the science of aeronautics, so remarkably rapid during recent years, has led to innumerable problems both of practical and of purely theoretical interest. Progress has tended in two sharply defined directions, the elucidation of the aerodynamical basis of flight dealing exclusively with the question of the interaction between machine and air, and the production of an engine which is at the same time strong, light and of sufficient Horse-Power to provide the necessary driving force. The first real and vital step in the solution of this latter problem lay in the discovery and perfection of the internal combustion engine whose future developments would appear to be confined almost entirely to refinements. Investigations of the aerodynamic basis of flight have however been so suggestive that even at this stage, when a vast amount of experimental data has been accumulated, the field of development instead of narrowing is rapidly widening.

§ **2.** That the machine may be capable of continuous and safe flight three main problems must be solved—the problem of sustentation, of propulsion and of stability. For purposes of classification, aircraft may be divided into two distinct types according to the method adopted to furnish the sustaining force. Lighter-than-air machines of the rigid and of the non-rigid types, depending for their support on the volume of air displaced by their inflated envelopes, have developed rapidly from the simple balloon to the modern Zeppelin, Schütte-Lanz and Kite balloon, so useful in modern warfare. Partly on account of their less cumbersome nature, much more rapid progress has, however, been made with heavier-than-air machines, depending for their sustentation, as will shortly become apparent, on their forward motion.

It is to a study of this latter class of craft that the present book will be devoted, although it will be clear that many of the principles outlined and developed will apply with equal force to both types. The principles determining the most suitable forms for struts and bodies, for example, may similarly be utilised in the selection of the best "stream line" shape for the envelope of an airship.

§ **3.** The sustentation required for the support of a heavier-than-air machine depends for its production on a simple elementary dynamical fact. It is a well-known experience that when a body is moved rapidly through a viscous fluid like the air, it encounters a resisting force, depending for its direction and magnitude on the shape of the body and on its speed. Only in certain cases, as for example when the body is symmetrical about the direction of motion, will this resisting force be in that direction.

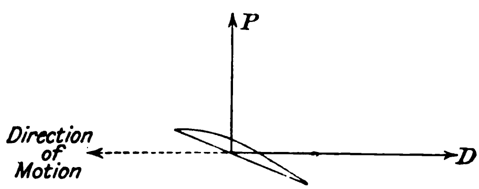


Fig. 1

Normally this resulting force which must be overcome to maintain the motion may be regarded as composed of two components, one D the dragging force along the direction of motion, and the other P in a plane perpendicular to that direction (fig. 1). If some external means be devised for continuously overcoming the dragging force D, and maintaining the motion, then P will continue to exist, and can be utilised wholly or partially as a supporting force provided it is acting upwards. The greater the ratio of P to D the more effective can this body be considered as a supporting surface, and it is one of the problems of aeronautical research to determine the best shape of such. It is the wings W, Plate I, of an aeroplane that chiefly supply the lifting force P, and the propeller or airscrew S that provides the means of propulsion by overcoming the dragging forces. The means of sustentation and propulsion go hand in hand.

§ **4.** Roughly speaking the propelling force is secured in the backward discharge by the airscrew or propeller of the air through which it moves, thus supplying itself with forward momentum sufficient to provide a forward force neutralising the drag. The exact mechanism whereby this is produced must be reserved for a later chapter, but the student will quickly recognise that on the same principle as that described above for a supporting surface, the rotation of the propeller blade about its axis will give rise, not only to a resisting force in the disc of the propeller, but also to a force perpendicular to this and therefore parallel to the axis. So long as the dragging forces resisting the rotational motion are overcome, so long will a propelling force be obtained. The power required to continue the rotational speed is derived from the engine *E*.

§ **5.** Although the two most important elements in an aeroplane, apart from the engine, are thus the wings or supporting surfaces and the propeller, safety in flight can only be secured by the addition in various positions of certain controlling and stabilising surfaces. The tail system *T* for example, consisting of a certain arrangement of relatively horizontal and vertical planes situated some distance in the rear of the main planes or wings, partially fulfils the function of imparting stability to the machine. Thus should an accidental pitch or tilt during flight be communicated to the aeroplane so that the tail rises, a restoring couple is immediately brought into play by the pressure of the wind upon the tail planes.

The rudder, *R*, a vertical component of the tail system, and controlled from the pilot's seat, is used in a similar way to the corresponding part of a boat, to alter the direction of the axis of the machine by the pressure of the wind upon it when turned from its normal axial position.

For certain purposes, to execute special motions such as turning during flight, which demands that the machine be banked or rolled about its longitudinal axis, *OX*, it is necessary to secure that the lifting force on that portion of the lifting surface to one side of the pilot be greater than that on his other side. This is obtained by altering the inclination of the wing flaps or ailerons, *F*, situated at the rear or trailing edges of the wings, thus practically changing the form of wing surface, with a consequent alteration in supporting force.

§ 6. There is one aspect of the problem of securing safety in flight which, in a sense, is vital for the determination of the relative arrangement of the parts. This deals with the introduction of certain otherwise extraneous members, existing almost purely to give strength to the construction. The principal among these are the struts, *C*, almost vertical wooden pillars binding the two wings, in the case of a biplane, securely together. There exist in addition for a like purpose numerous bracing wires, *B*. Connecting the main body of the machine with the tail system there is a long tapering construction known as the fuselage, *A*, built internally on the general lines of a Warren or N girder and covered with fabric, which as will shortly be seen reduces the resistance. Except in very particular cases such portions of the machine as owe their existence purely to requirements of strength, tend generally to increase the resistance of the whole, a distinct aerodynamic disadvantage. The greater the resistance the aeroplane encounters during flight the more power must the engine and propeller possess to overcome it.

It may be stated generally as a sound aerodynamic principle, that the efficiency of any machine will be increased by so shaping the parts exposed to the pressure of the air as to reduce this resistance to a minimum. From this point of view the fuselage, existing initially for constructional reasons, rather tends to operate as an aerodynamic advantage, for it will be brought out later that a long tapering structure of this type fitted behind the body actually reduces the resistance of the latter portion.

The body *D*, properly so called, is normally situated axially in the middle of the lower plane with the engine and the propeller, in front in the case of a tractor engine, and behind in the case of a pusher. It contains the pilot's and observer's seats, and frequently also space for a gun. All the instruments which the pilot makes use of for determining his height, speed, rate of climb, etc., are kept in this portion of the machine. In the case of a pusher the fuselage is of course replaced by an outrigger. It will shortly become apparent that the pusher type is much less efficient as a flying machine than a tractor, but it possesses a distinct advantage from the military point of view of providing a clear field in front for observation and the operation of the gun.

§ **7.** It has been explained that the lifting force on the wings depends upon the speed, and therefore a machine cannot rise from the ground until it has attained a certain minimum forward speed relative to the air. To secure such motion over the ground the aeroplane is usually fitted with a chassis or landing gear G consisting of two wheels on an axle connected to the body and lower wings by a system of springs and other shock absorbing devices. They must in addition be of sufficient strength to withstand the impact on the ground when landing, although when this manoeuvre is most efficiently executed, practically no shock should be experienced. Towards the rear under the tail there is usually placed a skid H whose function is to afford protection from possible damage to the tail on reaching the ground, while at the same time acting as a drag to retard the motion after landing.

The main portions of a machine described above are practically essential to every aeroplane, but do not necessarily occupy the same position or prominence in all. Aeroplanes are always constructed to fulfil special functions, but, while one may be built for high speed work, another for weight carrying power, the general form of construction is the same, the special characteristics demanded making themselves evident in the disposition, dimensions and shape of the parts.

§ **8.** It is the problem of aeronautical research to determine the particular method and details of construction of a machine to fulfil certain specified functions. There are in general two main methods of attack, distinct, but at the same time supplementary. In the first place mathematical and aerodynamical theory may be utilised to indicate where and how certain parts such as the tail system, for example, are to be placed in order that the machine may give a certain performance under given conditions. One of the best and at the same time most useful illustrations of this method finds its place in the problem of determining the position, dimensions and inclinations, of various portions to produce a machine, which is of itself naturally stable under the ordinary conditions of flight. Mathematical investigation has likewise furnished much interesting information regarding the nature of the air flow in the vicinity of certain of the parts upon which so much of the efficient performance depends. In the second place experimental investigation, at present far ahead of the mathematical theory, provides

a potent means of attack. For this purpose accurate methods would require to be devised for the determination and isolation of the effects of innumerable slight modifications in design.

We may imagine that a complete aeroplane, capable of flight, has been constructed, and suppose that instruments exist for accurate measurement of relative wind speed, ground speed, rate of climb, thrust of propeller, etc. Under these conditions the experimental method applied to determine the effect of some modification of shape of body, for example, would demand that the machine be fitted with different types and shapes of body and the speeds, thrust of propeller, etc., be obtained from a series of flights with each type of body. Generally speaking very useful information has been obtained under these conditions, but it is evident at the outset that certain limitations are imposed. The accuracy of such experiments and the validity of the results will be limited on the one hand by the variability of atmospheric conditions and temperaments of pilots, and on the other hand by the accuracy of the instruments at their disposal. It will shortly become evident that under certain circumstances errors due to unreliable instruments, vertical currents, etc., are necessarily involved in methods of measurement of aeroplane performance, and it is then only by an elaborate and extensive series of experiments that these can be allowed for or eliminated. Such inaccuracies frequently tend unfortunately to obscure the effects of small variations in design, only to be determined by the most delicate means at our disposal. The power of isolation of the effects of such small changes is, as already indicated, one of the main tests of successful experimental work of this nature, and the impossibility of attaining this under the conditions of full scale tests—tests on complete machines—compels us in these cases to seek some more manageable means of investigation. It must not be forgotten, however, that full scale tests, in their proper sphere, have their validity provided the conditions under which they are carried through are clearly understood.

§ 9. There is one line of development of experimental investigation which from the point of view here discussed has been highly successful in that it affords complete control, at the will of the experimenter, of the conditions under which the tests are carried through.

Models of manageable dimensions of a complete machine or any part of it can be constructed with extreme accuracy, and by placing these in artificial wind whose speed can be regulated at will, the ordinary conditions of flight can be imitated, and by suitable apparatus the forces brought into play on the whole machine, or on any arbitrarily selected part, can be measured. The delicate instruments used for this purpose will shortly be described. It need scarcely be remarked that as far as the air forces upon the machine are concerned, it is immaterial whether the latter be in actual flight through still atmosphere, or whether the air moves with the same speed past the machine—the air forces in fact depend purely on the relative speed of machine and air.

Model experimental work naturally separates itself into two distinct types, those cases in which the method is applied to determine the best *shapes* of parts, struts, aerofoils, etc., suitable for certain purposes, and those in which the actual *forces* on the complete machine or on its individual parts are required. It is clear that the results obtained for the first class by the comparison of various forms of section and contour of a special portion will in general be directly applicable to the full scale machine without modification. For the second class, however, a serious difficulty must be faced. A little consideration will show that no very simple law for obtaining the magnitude of the forces that would be brought into play on a full scale machine under flying conditions, from those measured on a comparatively small model under experimental conditions, can be framed. The problem thus encountered has given rise to the whole section of aeronautical theory known as "scale effect," and has for many practical purposes been successfully solved. The method of transition from model to full scale and the conditions under which wind channel experiments are legitimately applied to the full size machine will be discussed at some length in Chapter IV. A thorough understanding of this section of the subject is a necessity for a serious study of aeronautics, for it supplies the key to many otherwise complex aerodynamic phenomena.

§ 10. The normal type of wind channel (see Plate II) within which such experiments are conducted, as designed and used at the National Physical Laboratory, Teddington, consists in the

later type of a hollow horizontal wooden structure 80 feet long. This is composed of two parts. The working portion 7 feet square, extends from the mouth at which the artificial wind enters, to a distance of 45 feet, the sides of the channel being closed to the outside air except at the mouth. The channel widens out at the other end into a compartment, the walls, roof, and floor, of which consist of spaced laths so arranged as to allow of an outflow of the air from the channel to the room to cause the minimum disturbance to the general atmosphere. Forward in this compartment, with its shaft horizontal, is situated the propeller by whose rotation the air is drawn along the channel at a speed easily controlled by the regulation of the speed of revolution of the propeller. At both ends of the channel proper there are placed two metal honeycombs to steady the distribution of air speed across sections of the channel. The velocity of course increases from the mouth to the propeller but the rate of change is comparatively small and is in fact, as experience shows, often negligible in the working portion. Through one of the walls a small circular hole is bored and connected by a tube to a pressure gauge (to be described later). By measuring the pressure of the air as it moves past this position in the channel the wind speed can be determined with extreme accuracy and fluctuations from any proposed value can immediately be damped out by a corresponding regulation of the rotation of the propeller. Details of the method of calibration will be reserved for the next chapter after certain of the laws of air flow upon which the instruments depend have been discussed.

§ **11.** From what has been said it will readily be understood that the type of work undertaken by this method of research would demand an extremely delicate and accurate means of estimation of the forces brought into play. This has been achieved by the type of balance shown in skeleton in fig. 2 and Plate II devised and developed at the National Physical Laboratory.

In effect the instrument consists of an upright support PAM passing through the channel floor, the model to be tested being M supposed fixed for the moment in the requisite position relative to the wind. At A is an oil seal whose function is to prevent disturbances by air being drawn through into the channel from the outside. $APQRT$ is all one piece supported at the single point

P of the fixed standard *B*. *R* is a stability weight, by varying the position of which vertically, the sensitiveness of the balance may be adjusted.

From a knowledge of the dimensions of the parts, the wind force *F* acting on the model (supposed symmetrical about a

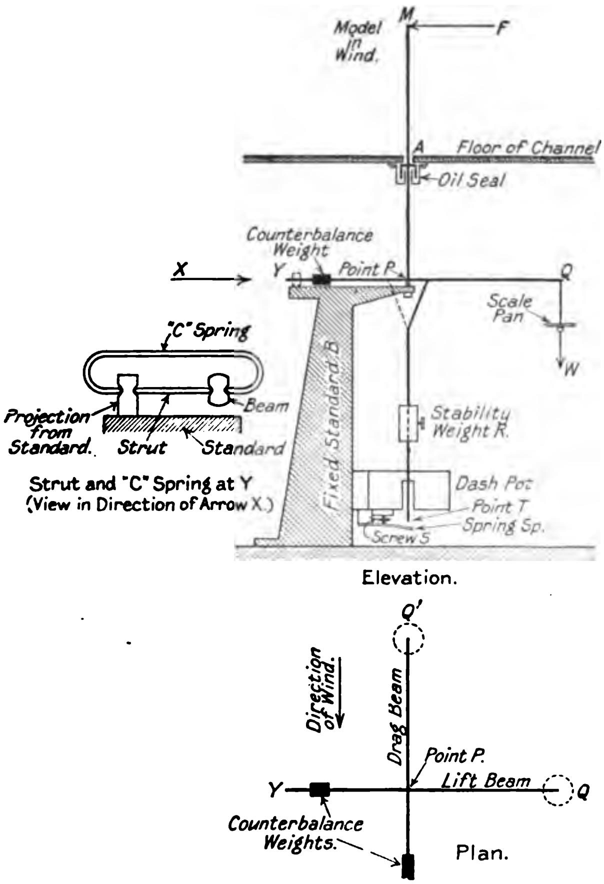


FIG. 2

horizontal plane through *F*) can easily be obtained by determining the weight *W* necessary to maintain the arm *PQ* horizontal. From fig. 2 it will be seen that the direction and magnitude of the resultant force in a horizontal plane will be known when the two components along and perpendicular to the wind direction

are found. For this purpose there exists another arm identical with PQ, but perpendicular to it in the horizontal plane. The forces on the two arms can then be determined separately and simultaneously. The balance is prevented from turning about the vertical axis under the influence of the moment of the wind forces on the model by constraining the arms by means of a strut and "C" spring shown in fig. 2, connecting the portion PY to the fixed standard. The importance of this will be evident from the fact that a slight turn through an angle of 1/20 of a degree would be sufficient so to alter the components measured as to involve an error in some cases even larger than the total normal experimental errors. For the measurement of the moment on the model about the vertical axis the strut and "C" spring are removed and replaced by a calibrated spring, and the screw at S loosened until the spring Sp bears upon the point T. The balance is now capable of rotation only about the vertical axis TP, and by measurement of the degree of compression or extension of the calibrated spring required to bring the balance to its normal position the moment can be determined. In actual practice the balance may undergo further modification in order to measure other forces and moments brought into play but the general principle is as described above, and this is the form most commonly used.

§ **12.** For certain classes of experimental investigation, where forces etc. are measured on a complete model, an additional balance is frequently brought into operation. As indicated in fig. 3 the tail of the model is hinged at M to the upright support PAM (fig. 2) while the forward portion is supported at the wings by two vertical wires to an overhead balance resting on the channel roof. The wires from the model passing over the two pulleys P are connected to the small winding drum, by the rotation of which any adjustment as regards angle of incidence may be made on the model. The wind forces brought into play are transmitted to the main channel balance through the hinge M and to the upper balance through the supporting wires the forces in which can be measured alternately or, if desired, simultaneously. The forces in the wires are measured by a balancing moment about the supporting points a and b applied at the scale pan. For research on complete models of machines this method is extremely valuable on account of its accuracy and rapidity.

Overhead Balance

Plan of Frame "A"

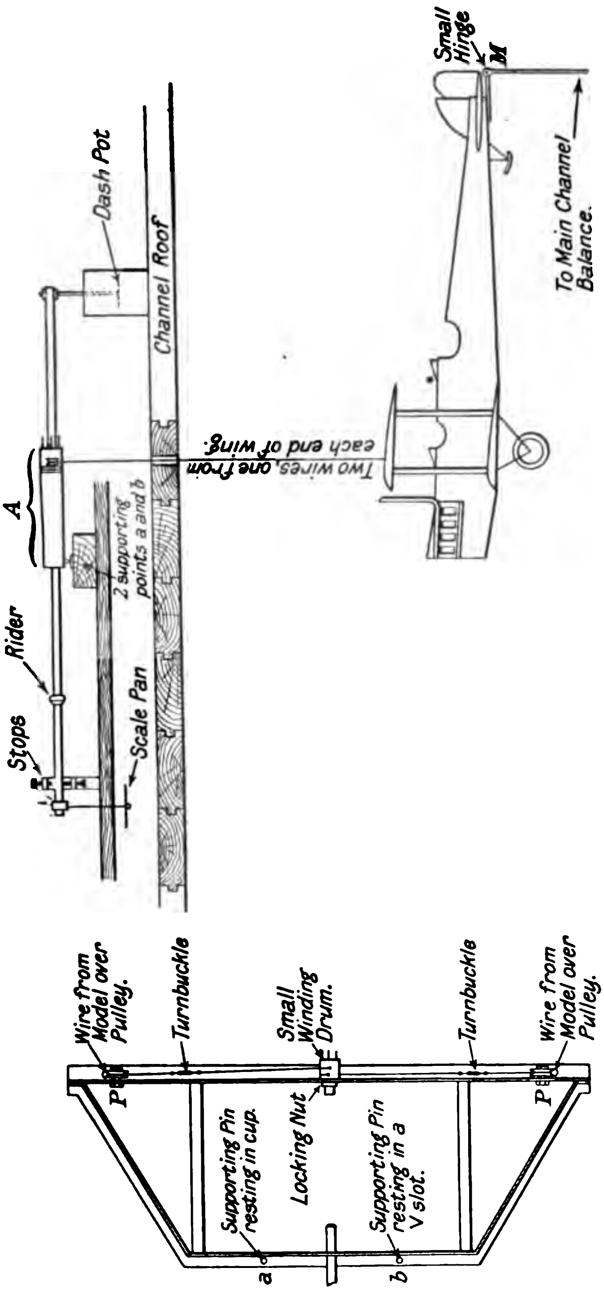


Fig. 3

§ **13.** Experience has shown that the instruments at our disposal for wind channel tests enable the forces and moments to be measured to an accuracy of approximately 1 %. This is extremely fortunate in view of the fact that frequently the effects of very delicate changes in relative positions of parts and modifications in shape require to be investigated. In consequence great care must be taken to ensure that the models themselves be made with extreme accuracy. It has been remarked, and the real significance will be brought out in the next chapter, that the nature of the air flow in the vicinity of any body determines the forces and pressures exerted by the wind upon it. If a strut, whose shape is such as to allow the air to stream easily past it, be not quite truly formed, the discrepancy from the perfect shape will continually act as a source of disturbance to the air by altering the direction of the flow in its neighbourhood. Energy will thus be uselessly expended and the resistance of the strut consequently increased. Certain portions of the strut—and in fact of most bodies entering into aeronautical work—may be more sensitive than others to such discrepancies in shape. It is the object of the next chapter to study the laws of air flow and the principles that must be used in the determination of the shapes of bodies most suitable for the functions demanded of them. The importance of this aspect of the subject cannot be too strongly emphasised.

PART I

CHAPTER II

FLUID MOTION; ELEMENTARY THEORY AND EXPERIMENT

§ **1.** It is commonly recognised that the impact of the molecules of a gas against the sides of any vessel enclosing it gives rise to a pressure whose magnitude at any point may be measured by the change of momentum experienced per second perpendicular to the surface by the molecules impinging on unit area. The pressure thus originated—and it operates not merely on the surface of the enclosing vessel but on the surface of any body immersed in the gas—may be regarded as a static pressure and in the case of a liquid is what is commonly understood as the hydrostatic pressure. If on the other hand the body be in motion, the particles of gas impinging on the surface will undergo changes in momenta that would not otherwise come into being if the body were at rest. On the surface consequently an additional pressure is experienced at each point frequently referred to as the dynamic pressure. If the effect of the body however were simply to cause a change of momentum perpendicular to the surface of impact, many of the complications that arise in aerodynamic analysis would be removed, but unfortunately the problem is made more intricate by the fact that the molecules composing the surface of the body and the molecules of the gas itself exert an attractive force on each other giving rise to a distinct surface effect.

§ **2.** To trace out and sum up the effects of the impacts of the individual molecules is clearly almost impossible, but this continual sweeping of the particles to and fro in all directions produces in the gas as a whole three distinctive properties which may be regarded in a sense as definitive of the medium and from which it will be possible to proceed to a more detailed discussion of the laws governing the motion of the gas and the pressures which it originates.

It is clear that the sum of the effects due to the rapid series of impacts may be regarded really as an average effect and it is from this point of view that the question will be approached. The particles at any instant enclosed in any small region have each their own velocities at that instant, characteristic of these individual molecules. There will, however, exist over the region an average velocity, average both in magnitude and direction. It is proposed for the present purpose to replace this gas composed of discrete particles by a homogeneous, non-molecular medium, such that the velocity of any small region is equal in magnitude and direction to the average velocity as explained above of the corresponding region of the real gas. The solid boundaries enclosing this equivalent fluid are taken as identical with those in the original gas. It will be assumed that if this homogeneous fluid possess properties sufficiently definitive of the original gas, from the present point of view, then the resultant pressures and resulting motions will be the same as those experienced on the average in the original—the average extending over a very short time.

§ **3.** For a true understanding of the laws governing this hypothetical fluid, it is necessary to investigate those properties of the gas which in any way effect and determine its action. In the first place, each molecule possesses its own distinctive mass which will enter as a factor in the transference of momentum from one molecule to another. The total mass per unit volume of the molecules included in a small region surrounding any point is then understood as the density of the gas at that point. It will be supposed that this is also the density of the homogeneous fluid existing at the corresponding point.

Under the influence of compressive forces, the volume of a gas may be reduced, and, since pressures are normally brought into existence during the motion, the property of compressibility must play its part. It is assumed that the compressibility at each point of the homogeneous medium is equal to that over a small region enclosing the corresponding point in the real gas. There remains yet one other property which must be transferred to the fluid to make the equivalence complete.

§ **4.** Suppose the gas as a whole to be moving steadily in any region in the direction AB and CD, fig. 4, with mean velocities $V + \delta V$ and V respectively where AB and CD are a distance δn

normally apart. The molecules, shooting about in all directions, will be continually effecting a transfer of momentum from one portion of the gas to another, but on the whole during any short interval of time gas molecules will not tend to accumulate more in one region than in the other. If PQ therefore be a small portion of any surface situated between AB and CD and parallel to them, the number of molecules passing downwards over PQ in unit time must equal the number passing upwards, their masses being supposed all equal. The gas, however, in the upper layer moving with higher speed than that in the lower, will, by this process, contribute to the latter more momentum per unit time than it sends back. There is accordingly a rate of change of momentum over this intermediate surface in the direction PQ. This is equivalent to the statement that there exists at the boundary between the two layers, a dragging force on the upper and an accelerating force on the lower. It would appear that wherever in a gas there exists a velocity gradient there must exist also on intermediate surfaces this dragging force, the slower moving layers retarding the motion of their more rapidly moving neighbours. This tangential force is usually termed the force of viscosity.

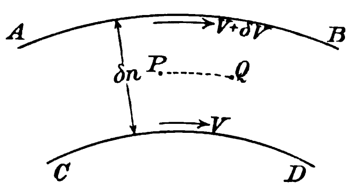


FIG. 4

It is now no difficult matter to translate this conception into an assumed property of the hypothetical homogeneous fluid whose equivalence it is desired to establish. It is only necessary to assume that this fluid is capable of withstanding a shear depending for its magnitude on the velocity gradient according to the law existing in the gas. Actually it is found by experiment in the case of any fluid that this viscous force is proportional to dV/dn. The coefficient of viscosity μ as understood in our homogeneous fluid will be the viscous force exerted on the layer of fluid of unit velocity at unit distance from a fixed plane at which the velocity is zero. Maxwell obtained the value of μ by vibrating a horizontal circular disc suspended by a wire about its axis, close to a fixed parallel disc. The layer of gas between the two was accordingly undergoing shear, and the original vibrations were gradually damped out by the viscous resistance of the gas; μ can evidently be calculated from the damping.

§ **5.** Before coming to a discussion of the types of motion of which the homogeneous fluid is capable, it will be necessary to determine what is the nature of the reaction between a solid boundary and the layer of fluid in immediate contact with it. The reaction between layers of the fluid themselves has already been considered.

We may imagine that in the passage of a body through a gas continuous intermingling of the molecules of the latter with those of the body more or less loosely connected with the surface takes place and that to some extent the gas may be imagined as partially penetrating for a very short distance into the body. The effect of this intermingling can be conceived as being an attempt on the part of nature to prevent a discontinuous change in density in the passage from body to gas. But an assumption of this nature involves as a natural conclusion that the motion of the fluid and body varies also continuously from body to gas. Experiments conducted to investigate the velocity of air and of water at the surface relative to a body moving within it, indicates that at any rate for moderately slow speeds no "slip" occurs at the surface. For higher speeds, however, the evidence on this point is not sufficiently conclusive to allow a definite assertion to be made.

§ **6.** The problem of theoretical aerodynamics, in the light of the present discussion, reduces itself to a consideration of the motion of a homogeneous fluid of given density, viscosity and compressibility: It is evident that certain of these factors will play a greater or a less important part in any problem according to the circumstances. If, for example, the motion be such that no large pressures be produced, compressibility effects will not become apparent and the forces and motions will be due almost entirely to density and viscosity. Numerous experiments have brought out the remarkable result that up to and far beyond the ordinary speeds of aeroplane flight the pressures which come into play are not sufficiently great to produce on the air any appreciable compression. It is only when the speed of flight approaches that of the speed of sound waves, that the air in the neighbourhood of the moving body undergoes compression. That being so it will be legitimate to introduce at once a simplification into the conception of the hypothetical fluid. It will be assumed incompressible, and there will remain only the two properties,

density and viscosity, with the condition of zero velocity at the given boundaries to affect the motion.

A complete mathematical development of the subject based on these conceptions of the fluid as foundation is of considerable complexity, and has not for that reason furnished very much information. Fortunately however the path of experimental development has not been so retarded, for in water we possess a fluid to all intents and purposes homogeneous, dense, and viscous, and whose motion may fortunately be made visible. It is apparent that the type of motion obtained in the case of a body moving through water will not be identical with that obtained from motion through air on account of the different values for the densities and viscosities, but the general type of motion will be the same. It is possible in fact by a comparatively simple calculation to make the two cases exactly comparable. The details and interpretation of this calculation will be found in Chapter IV, and cannot be entered into at this early stage.

§ 7. Although the mathematical treatment of the motion of a viscous fluid is exceedingly difficult and intricate, the general nature of the flow that must arise under certain circumstances is plainly discernable from comparatively elementary considerations.

Two methods of considering the question at once present themselves. We may, in the first place, trace out the path and motion through all time of a particular small element of the fluid, following it in its career from point to point. If the motion at any point varies from instant to instant, the path thus traced out will not necessarily be the same as that followed by the succeeding small element. This will only be the case when the type of motion is independent of time, that is to say when the motion is steady.

A much clearer conception can however be obtained by following an alternative method. Imagine at any instant that the whole field of motion is covered by a system of lines traced out from point to point whose direction is everywhere that of the motion of the fluid. Such lines will be termed *lines of motion*, or *stream lines* showing as they do the direction of streaming at the instant. If the motion is steady this system is fixed for all time, and they give the actual paths traced out by the elements of the

fluid. In the case of unsteady motion, however, they vary from instant to instant and can only return on themselves if the motion be periodic.

§ 8. Consider what takes place in the neighbourhood of the edges of a flat plate AB placed perpendicular to the direction of the stream. Along AB itself the fluid sticks to the surface and is thus not in motion. The stream line AC which just misses the edge would normally in virtue of the inertia of the motion of the elements be shot off along the direction AC'. Its proximity to the body and the reduction in motion caused by the retardation of the layers of fluid in its immediate neighbourhood, however, cause the particles to tend to move along a path of smaller radius of curvature and the stream line bends inwards along AC. Similar causes operating drag the particles further and further inwards and a form of whirlpool or eddy is produced just beyond the edge. Actually the process by which this eddy is produced is not nearly so simple as that indicated, but that they actually do exist and materially affect the nature of the forces brought into play is well brought out by experiment. It is a familiar enough experience to produce two dimples on the surface of a liquid by gently drawing a teaspoon through it. One such eddy is usually produced at each edge of the obstruction. These rapidly increase in intensity, gradually drawing away from the body by moving down stream, and are finally totally disengaged from it. A new pair immediately begins to make its appearance and likewise moves off, and the process repeats itself ad infinitum. Numerous investigations have been conducted for the purpose of analysing this motion in its various stages, as the process by which it originates is not clearly understood. It has been found that there are two possible types of motion not however equally stable in the sense that if once established it will persist. Fig. 5 shows diagrams of the two types of motion, where in the one case the eddies are produced first at the one edge, and then at the other, so that on leaving the body they form two trails of vortices in an alternate arrangement, and in the second case where they are produced simultaneously at both edges. The conditions determining which of these types of motion will be produced in any case have not yet been discovered, but conditions of symmetry would indicate that the second type could only fail to be produced if some non-symmetrical disturbance had

entered. On account of this fact the first type is more usually obtained. On tracing out the later history of these eddies it is found that on moving down the fluid their energy, instead of remaining concentrated in the immediate vicinity of the very small regions of rotation, gradually becomes dissipated through the fluid by the viscous dragging forces of the neighbouring layers, and finally they lose their identity as definitely visible centres of rotation. It is not difficult to see that these eddies are continually carrying off considerable quantities of energy, which they have accumulated in their motion past the body, and must therefore

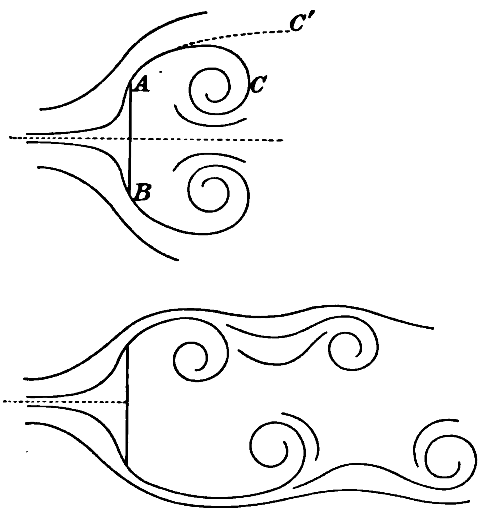


FIG. 5

make themselves felt in some manner on the resistance experienced by the body. Generally, if the motion is unsteady, so that the energy given up to the fluid in the flow past the body varies from second to second, so also will the force exerted. In the case here considered if the speed at a great distance be steady, the eddies are produced with a regular period and the force on the body becomes likewise periodic. At higher speeds, however, the period is so small, the fluctuations so rapid, and the variations in periodic force also so small, that the forces measured, to all practical purposes, become steady. It has been shown by theory, and confirmed by experiment, that the type of motion here

discussed does not come into being for a plane of given size until the velocity of the fluid at a considerable distance ahead of the body attains a certain magnitude. At lower speeds than this, eddies are apparently not produced. Curiously enough if the dimensions of the plate be doubled the flow changes from the steady type to the unsteady type at a speed of half the original value. It is found in fact, that for a body of given shape there exists a definite value of $v \times l$ where v is the velocity of the stream far ahead of the body, and l some fixed dimension of the body, say its breadth, at which the flow changes from the steady to the unsteady state. This is known as the critical value of vl. There is an extension of this conception which states that for two bodies of the same shape successive pictures of the flow past them are identical for the same value of vl. This is a particular case of the law of dynamical similarity which will be discussed in Chapter IV.

It is not merely when a sharp edge exists on a body that eddies are produced in a fluid. Any body in fact, if the value of vl be great enough, will set up trails of eddies beyond it, but the rate at which these are produced and their intensity and consequently also the rate at which they carry off energy will depend on the exact shape of the body.

Experimentally it has been shown that attempts to alter the shape by smoothing out sharp corners, or indeed any rapid change in curvature, result in the production of a state of flow in which the eddying region becomes narrowed down to a comparatively thin trail behind, with the intensity of the eddying and turbulence in that region considerably diminished (cf. figs. 6 and 7*, 16 *a* and 16 *b*). In addition to the effect of the removal of these rapid changes in curvature, it is found that an alteration in the length and shape of the hindmost part of the body frequently results in a steadying of the motion. If the body is short the fluid is capable of flowing at once in the form of an eddy immediately behind it, whereas, if the "tail" is long so that some considerable distance must be passed before an eddy can form properly, the whole nature of the flow becomes considerably steadied and less energy is thus dissipated, see figs. 6 and 7*, but it will shortly be seen that the dragging effect on the surface imposes a limitation to the extension of this principle. It would appear therefore, contrary to what might otherwise be expected, that low resistance of a body

* *Adv. Committee Reports*, 1912–13.

De Havilland (reversed)

FIG. 6

Beta

FIG. 7

De Havilland

FIG. 8

can be obtained by the presence of a blunted "nose" and a sharp "tail." Figs. 6, 7 and 8* show photographs of the types of motion past a series of struts where the effect of the tapering tail and the smooth contour of the surface is easily traced in the narrowness of the region of turbulence behind. Such bodies are extremely sensitive to slight variations in shape, a small discrepancy from the good "stream line form" acting continually as a source of disturbance to the smoothness of the flow, and consequently dissipating energy. Experiments conducted in the wind channels at the National Physical Laboratory have shown that the best shape of struts designed to give low resistance have approximately a fineness ratio of 3 : 1 or 4 : 1, the fineness ratio being the ratio of the length of the cross section to the maximum thickness. The foregoing remarks apply with equal force to the determination of the best shape of the envelope of a dirigible, and in fact to the determination of the contour of any surface in contact with the air in its flight, among which may be mentioned aeroplane bodies and wires.

§ **9.** In practice, it is usually difficult to set up the struts of an aeroplane accurately along the direction of flight, and as such bodies are frequently very sensitive to changes in angle of yaw, i.e. horizontal deviation from the direction of flight, the resistance might increase rapidly under such conditions. It is consequently regarded as desirable to select as the criterion for a suitable strut, not that the axial resistance is a minimum among a set, but that it is small and does not vary rapidly with angle of yaw. This latter remark applies with special force to the case of aeroplane wires. Fig. 9 shows the results of tests for a set of different wires where the resistance coefficient (i.e. the resistance in lbs. per foot length divided by $\rho l v^2$, where ρ is the density of the air, l^2 the area of cross section in sq. ft. and v the speed of the wind in ft./sec.) is plotted against angle of yaw. Wire A whose resistance is low, and variation over a large range of angle of yaw small, is evidently much more suitable than the others for that purpose. Apart from this, this wire possesses the special advantage of ease of manufacture.

§ **10.** Very recent experimental work has indicated the existence of certain curious and suggestive phenomena regarding

* *Adv. Committee Reports*, 1912–13.

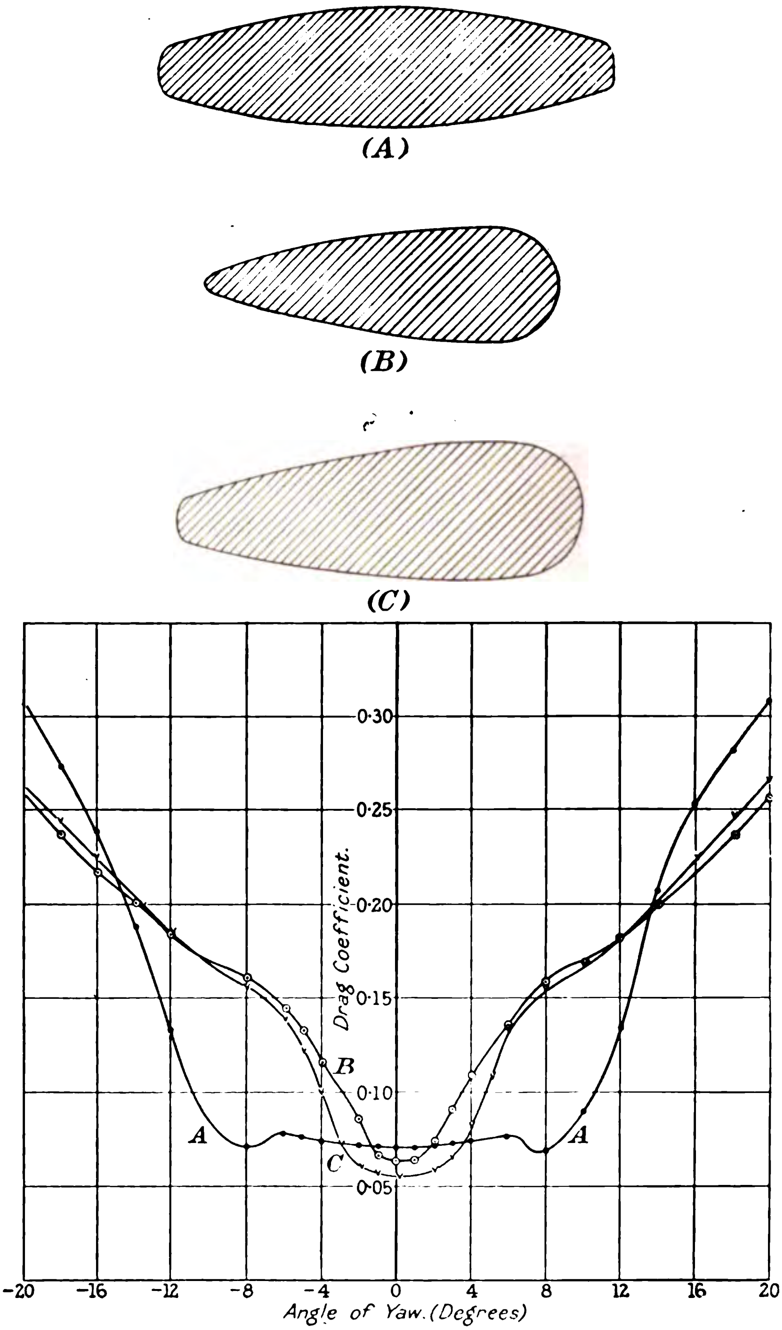
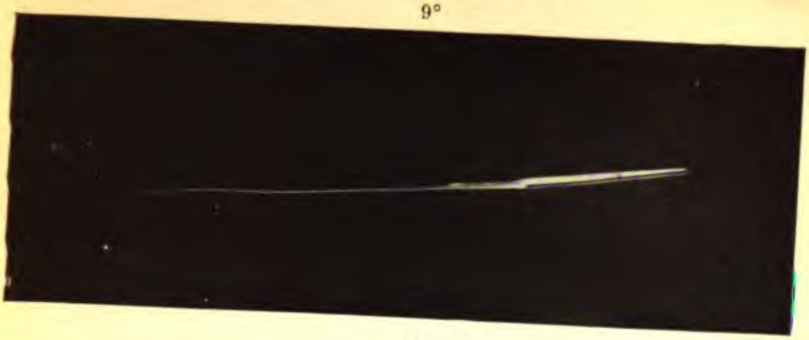

(A)
(B)
(C)
Drag Coefficient.
0·30
0·25
0·20
0·15
0·10
0·05
A
B
C
A
-20
-16
-12
-8
-4
0
4
8
12
16
20
Angle of Yaw. (Degrees)

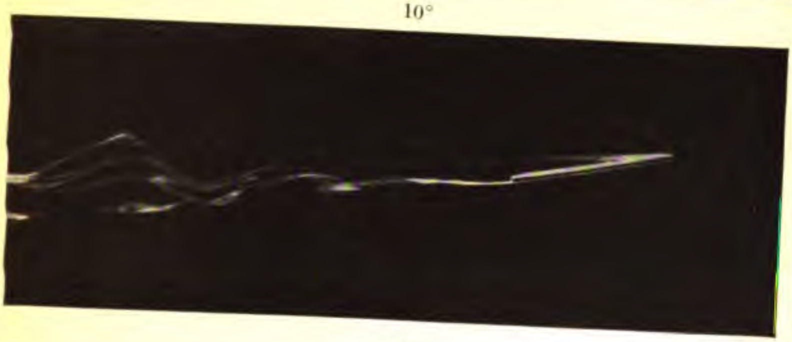
Fig. 9

FLOW PAST FLAT PLATES

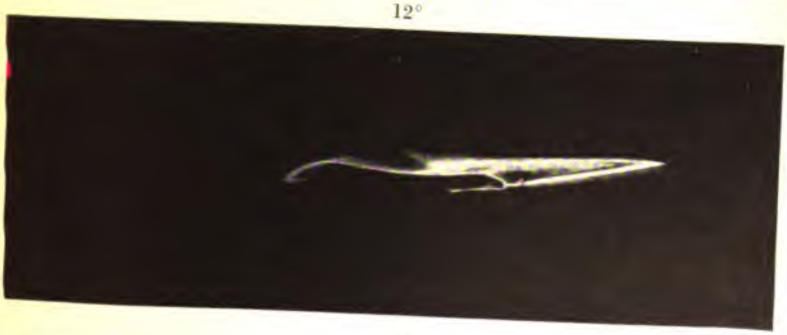
9°



10°



12°



15°

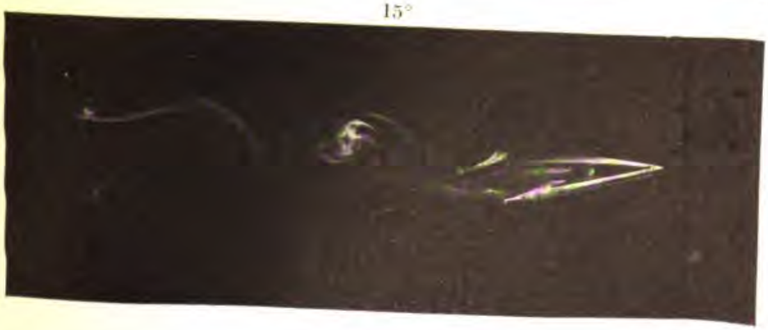


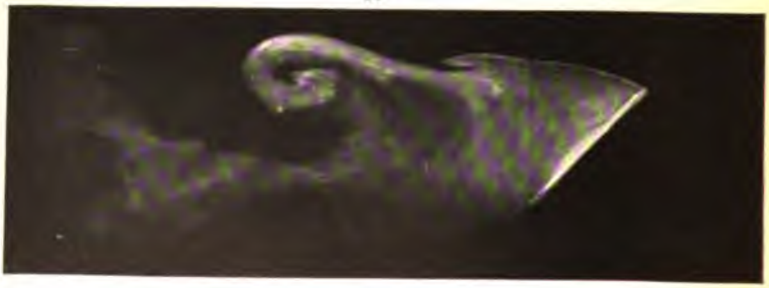
Fig. 10

FLOW PAST FLAT PLATES

20°

25°

45°



90°

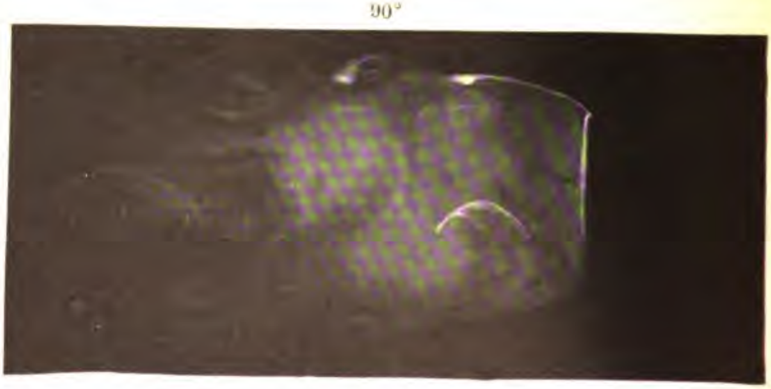


FIG. 11

the air flow in the neighbourhood of a strut. It has been found that for certain bodies of strut form situated in a wind of given speed, the resistance experienced, although approximately proportional to the square of the speed, is not unique but may have two or even three distinct values. By adequate control of the experimental conditions, as for example by raising or by decreasing the wind speed from a lower or from a higher value, to that required for the test, any one of these resistances may be obtained at the discretion of the experimenter. In view of the fact that the resistance experienced depends purely on the nature of the air flow, this result would seem to suggest that with given wind speed and shape of body the type of flow is not uniquely determined, but that the nature of the disturbance or previous history of the air exerts some selective effect.

The problem is too complicated to enter into here, it is not in fact clearly understood, but its solution evidently involving the question of the stability of fluid motion would throw a very strong light upon the general elucidation of aerodynamic phenomena.

§ 11. From the point of view of experiment, the aerofoil presents one of the more complicated problems in aerodynamics. The possible variation in shape, attitude to the wind, speed, etc., are so numerous that an elaborate series of investigations is necessary before accurate knowledge of the conditions governing the forces brought into play can be derived. It is to experimental work in the wind channel that one must turn for an accumulation of knowledge, regarding the magnitude of the forces, but on account of the invisibility of the air, experiments in a water channel are much more suitable as a means of discovering the laws of the fluid upon which the production of these forces is based. Experiments on flat plates in a uniform stream show the existence of two trails of eddies or vortices, one starting off from each edge (see figs. 10 and 11). If instead of a flat plate an aerofoil or indeed any curved surface be placed in the stream at a comparatively low angle of incidence, the eddying region becomes considerably narrower, and viewed from the point of view adopted when considering struts this fact would indicate that bending or cambering of surfaces would involve a distinct improvement in resistance. But resistance is not by any means the only factor to be considered.

It is to the vertical force exerted by the fluid on the aerofoil, that one must look for providing the means of support to the machine, and fortunately the cambering of the wings not merely diminishes the resistance under certain circumstances, but increases the lifting force by inclining the resultant upwards.

Experiments which will shortly be described on the pressure at each point of such a surface indicate, as shown in fig. 12, that, on the under surface, the pressure is everywhere positive, this region extending round the nose to a point on the upper surface a short distance in front of the maximum ordinate. From there onwards along the upper surface the pressure is negative, that is to say less than the atmospheric pressure. The maximum positive pressure, found experimentally in all cases to be $\frac{1}{2}\rho v^2$, occurs just at the nose while the maximum negative pressure occurs just before the maximum ordinate a. This pressure, experimentally measured, is the component perpendicular to the surface at the point. The upper surface accordingly contributes towards the lift

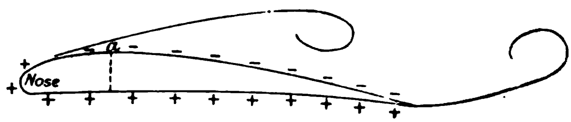


FIG. 12

by a suctional effect, while the under surface by a direct positive pressure. A comparison between the magnitudes of these pressures shows however that the former is of considerably greater consequence in this respect than the latter, and since the maximum negative pressure occurs at a position of the surface almost horizontal it is clear that the position and magnitude of the maximum ordinate will be a factor of extreme importance in deciding the performance of an aerofoil. Experimental work showing the modifications produced in the actual flow by changes in shape has not been carried sufficiently far to justify conclusions being based upon it, but interesting information is forthcoming by an analysis of the changes in the flow caused by changes in the angle of incidence or attitude to the stream. Figs. 13, 14 and 15 provide such a set of photographs. From what has already been said in the case of struts it is not surprising to find that as the angle of incidence is gradually increased the flow suddenly changes

FLOW PAST AEROFOILS

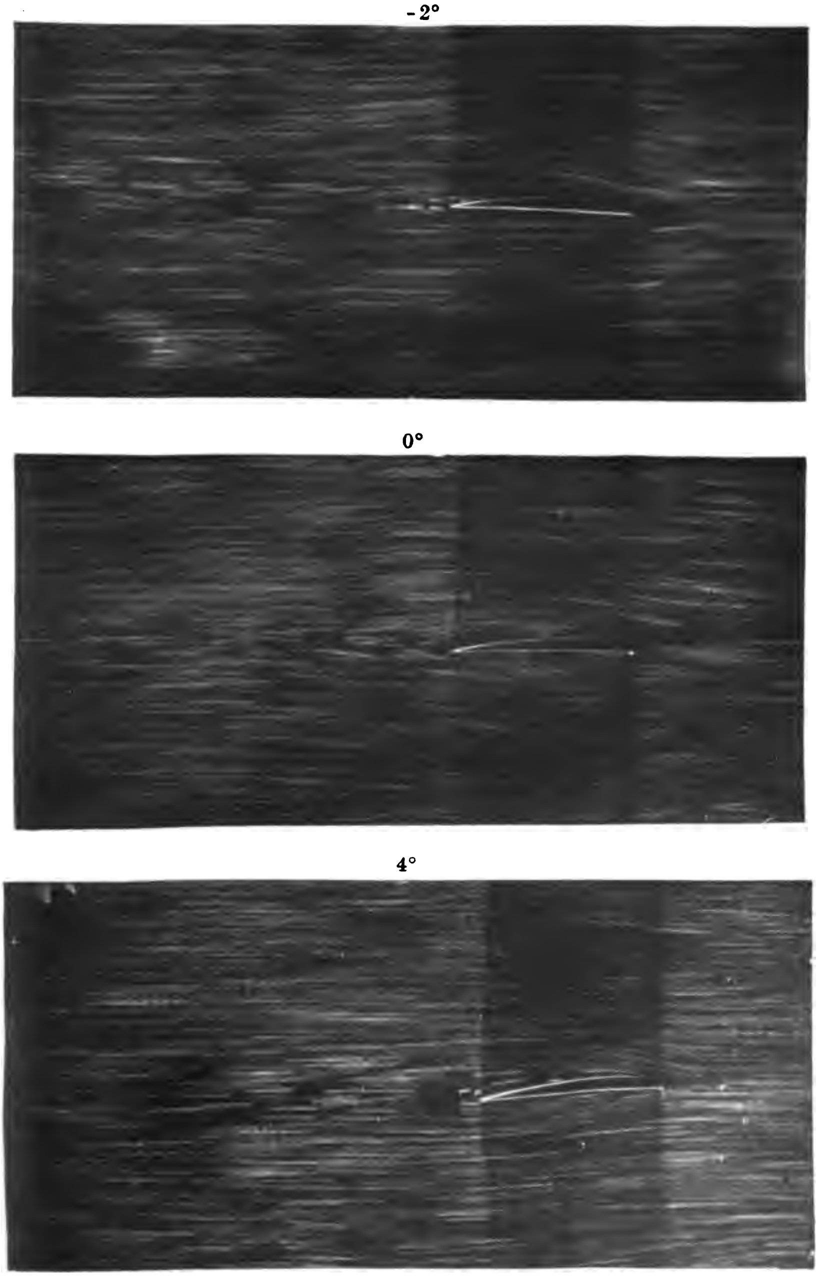
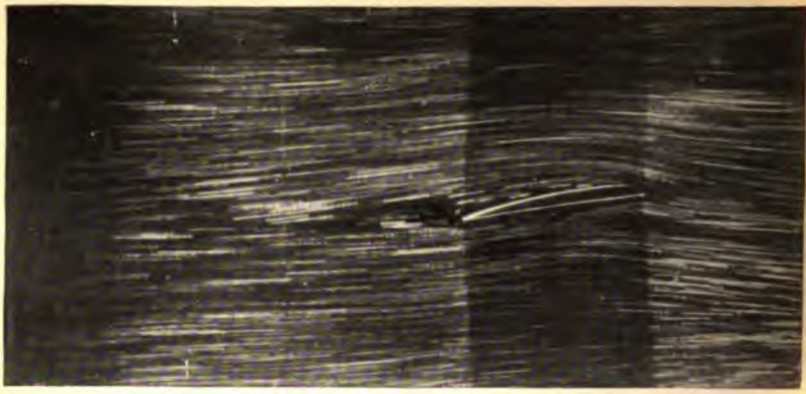


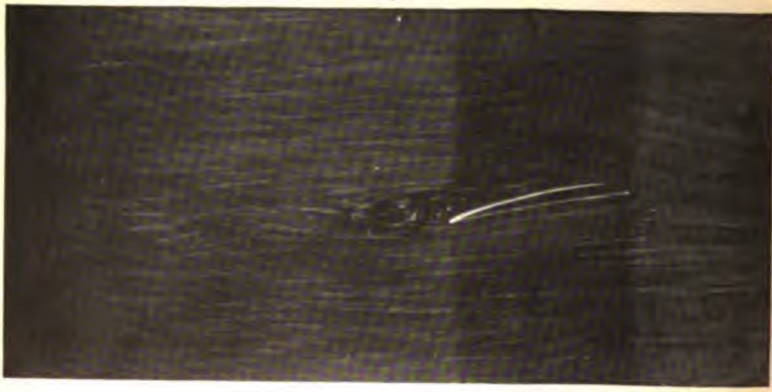
Fig. 13

FLOW PAST AEROFOILS

8°



10°

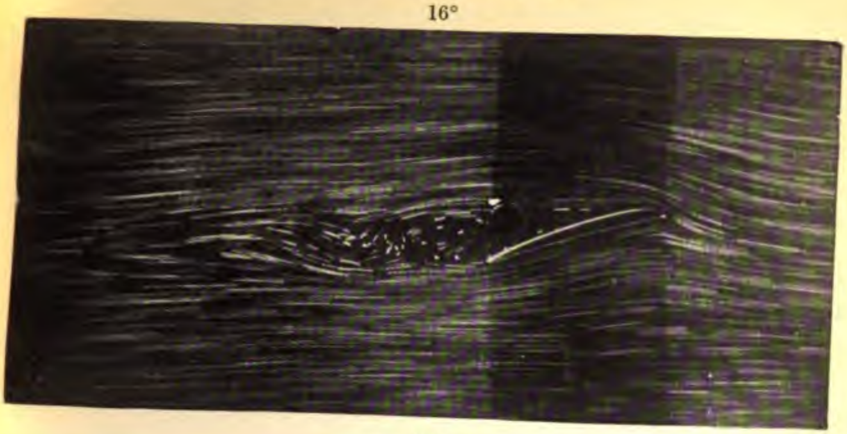


14°

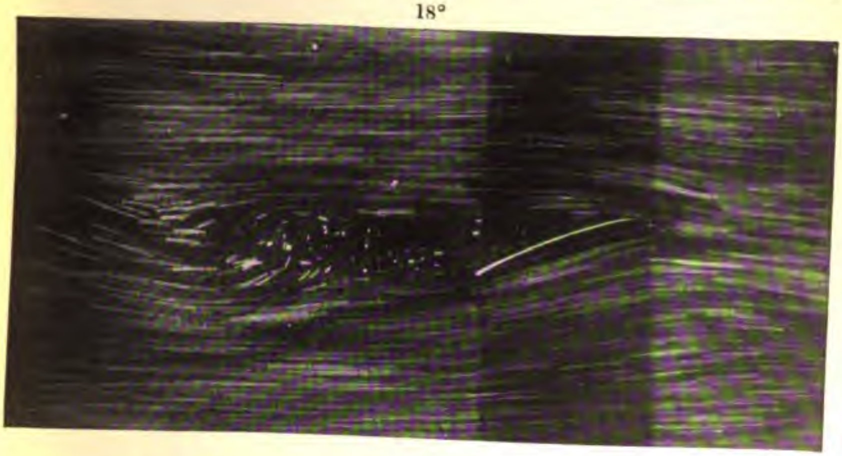
FIG. 14

FLOW PAST AEROFOILS

16°



18°



20°

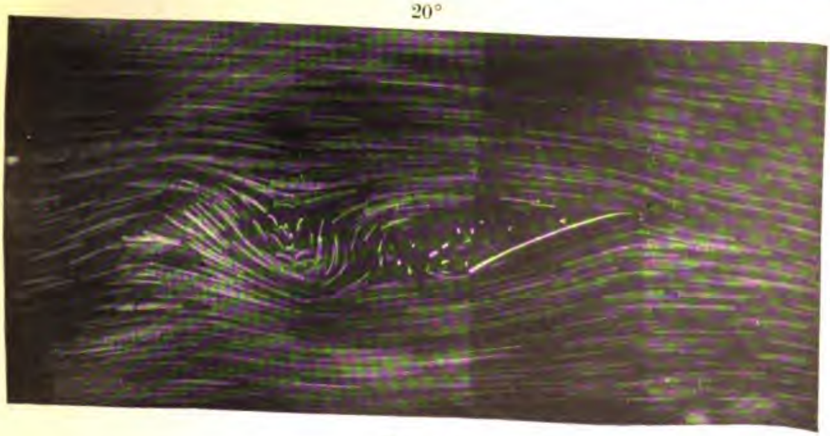
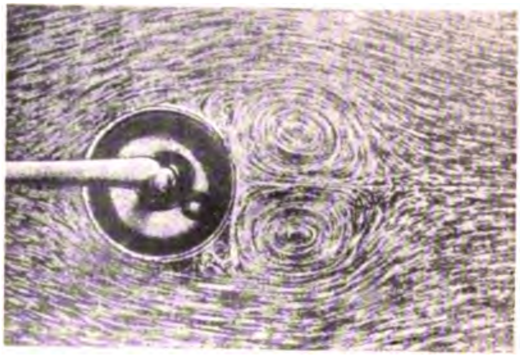
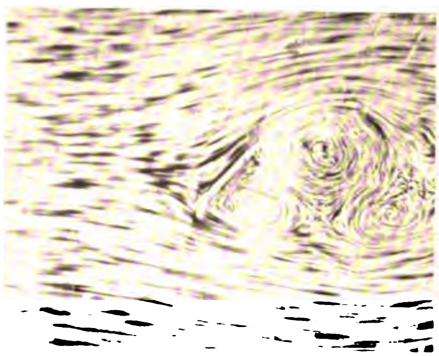


Fig. 15



Flow past circular cylinder

Fig. 16 *a*



Flow past flat plate

Fig. 16 *b*

from one of comparatively uniform eddying motion to a violent turbulence. What is in effect taking place will be understood when it is realised that for each body of given shape there exists one or more critical velocities, in the widest sense of the term, at which the nature of the motion undergoes a rapid alteration, a phenomenon already found to exist in the case of struts. The same aerofoil, by an alteration of attitude to the stream, is in reality equivalent to a body of different shape in relation to the stream, and a continuous alteration in angle of incidence appears thus as a continuous variation in shape of body, until finally an angle of incidence is reached at which the shape of body thus presented to the stream possesses that velocity as its critical. The word is here used in the sense that in the region of that speed the nature of the flow clearly undergoes a rapid change of type. The change is not of course quite sudden but it is comparatively sharp. The real cause of this rapid transition will not naturally be clearly understood until considerably more progress has been made in the theory of fluid motion, but the fact of its occurrence is of vital consequence for a thoroughly reliable anticipation of the performance of an aerofoil. The effect of this so-called "burble" point not unnaturally becomes apparent when the measurement of the forces brought into play on an aerofoil are undertaken in the wind channel. At the burble point the lifting force drops sharply and just as quickly rises again. Accurate water channel work of this nature could be a valuable asset to a clear comprehension of innumerable problems that present themselves in aeronautics, but the subject is unfortunately still in its infancy, and no sound and complete analysis of even the simplest case of the flow of a viscous fluid has yet been produced.

§ **12.** Reference has been made to the distribution of pressure over the surface of an aerofoil when situated in a steady current of air. When a body is immersed in a fluid at rest, the pressure at any point is perpendicular to the surface, but this is not the case when the fluid is in motion except in the ideal perfect fluid from which viscosity is excluded, for then the fluid is assumed incapable of exerting a tangential dragging force. From the mere fact that the sliding of the one layer of a viscous fluid over the other involves the existence of a tangential dragging force, the pressure at any point of a body immersed in it cannot in general

be perpendicular to the surface, but must be inclined to it. For purposes of measurement the pressure is usually transmitted from a pinhole through the surface, by means of a tube in the interior of the body to a form of manometer outside to be described shortly. Now it is clear that this method will only determine the component of the pressure at the point normal to the surface, and cannot measure the tangential or "skin frictional" component. The difference between the actual force on a body as measured in the wind channel and that obtained by integration of the normal pressures will thus provide an estimate of the value of the skin friction. Experiments conducted with the express purpose of determining the magnitude of this effect by Zahm and others have indicated that it is roughly proportional to the product of the area of surface exposed, and the velocity raised to the power 1·89. Remembering that the fluid in immediate contact with the body is at relative rest so that very small grooves in the surface are filled up, it is not unexpected to find that the skin frictional resistance is not appreciably affected by roughness of surface.

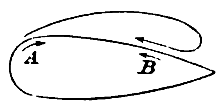


FIG. 17

The direction of each component of this resistance will of course be in the direction of the relative motion of the fluid near the surface layer, and thus in general contributes towards an increase in resistance, as at A, but in certain cases of struts, as for example where the eddying causes a relative motion near parts of the surface in opposite direction to that of the general fluid, as at B, the skin friction may actually contribute towards a diminution in resistance. (See fig. 17.)

The foregoing remarks on skin friction are typical of the light thrown upon the problem of resistance by a discussion of the pressure distribution over the surface of the body. A considerable amount of experimental work of this nature has been carried out at the National Physical Laboratory and much useful information obtained. In fig. 18 the variation in pressure over the surface of a model of a Parseval is shown. The pressure at the fore-part has never been found to differ appreciably from $+\frac{1}{2}\rho v^2$. For a short distance backwards from this point the pressure remains positive, rapidly falling to zero and then remaining negative. With some models the pressure again changes sign towards the tail. How each part of the body contributes towards the resistance becomes

immediately clear from an analysis of this curve. From A to B the pressure is positive and provides a direct resistance. From C to D the pressure is also positive, and since the inclination of the surface is of opposite sign to that at AB this tends to push the body forward and diminish the resistance. Along BC where the inclination of the surface is small the pressure scarcely affects the resistance at all. The advantage of having a long tail is at once apparent.

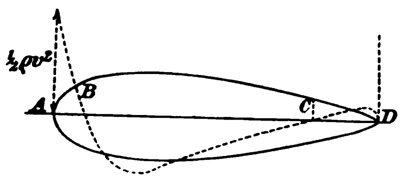


FIG. 18

§ **13.** In all cases where an investigation of the pressure distribution has been carried through it has been found, as already mentioned, that the maximum positive pressure always occurs at some point situated on the fore-part of the body, the value being always $\frac{1}{2}\rho v^2$ where ρ is the density of the air and v the velocity. If this point were regarded as the position at which a stream line meets the body and comes to an end upon it, it can be shown theoretically that if the fluid be further supposed devoid of viscosity, the pressure at that point must reach this same value. For a proof of this the student is referred to the next chapter.

The fact that the pressure at a point of a body meeting the air full on is $\frac{1}{2}\rho v^2$ has been utilised for devising an instrument to measure wind speed, known as the "Pitot Tube." The general outline is given in fig. 19. It is comprised of two hollow tubes fitting one inside the other, and closed along the circular edge ab. The outer tube is perforated at the points C, D, E, etc., and open again at the outside at F. The inner tube is open both at the front ab and at the end G. The fore-part XY is placed parallel to the wind W whose speed it is required to measure. The pressure across the section ab in the inner tube is then the ordinary dynamic pressure already mentioned $\frac{1}{2}\rho v^2$ added to the general atmospheric pressure say p_0. This total pressure $p_0 + \frac{1}{2}\rho v^2$ is transmitted through the end G to one arm of a pressure gauge of U tube type. At the perforations C, D, E, etc., at which the surface is parallel to the stream, there is no dynamic pressure since it does not meet the air directly but there exists the same general atmospheric

pressure p_0. This pressure is transmitted through F to the other arm of the U tube. The weight of the column of liquid per unit area between the two levels of the fluid in the U tube

$$= (\tfrac{1}{2}\rho v^2 + p_0) - p_0 = \tfrac{1}{2}\rho v^2,$$

and from a measurement of this difference in level the velocity can evidently be calculated. The verification of the assumption that the dynamic pressure at the nozzle is $\frac{1}{2}\rho v^2$ is usually effected by fixing the pitot tube at the extreme end of a whirling arm, consisting of a long horizontal beam capable of rotating in a horizontal plane at a speed of known amount. The pitot is so placed that the portion XY points along the direction of the relative wind, and the pressures are then carried as before from

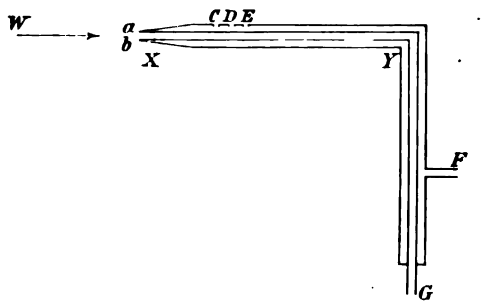


FIG. 19

each of the two projecting portions F and G to a pressure gauge as already explained. The difference in pressure thus measured is shown to equal $\frac{1}{2}\rho v^2$. The pressures set up in the two connecting tubes running along the arm to the gauge due to the centrifugal force of rotation of the columns of air enclosed of course balance each other when connected to the pressure gauge.

The pressure gauge, essentially of the U tube type, fig. 20 consists of two cups A and B to which the two tubes of the pitot are connected. From the two cups there is one continuous fluid system through the tubes CD, EF. The fluid in the two cups is the same (brine 1·06 density), but the bulb enclosing the end E is half filled by a dissimilar fluid (castor oil) of less density so that it floats above, and of such a nature as to form a meniscus at E where the brine and castor oil are in contact. The exact position of this meniscus in the portion CG will be determined by the difference in level of the surfaces in A and B and the

pressures in these cups. When the gauge is connected to the pitot, the air pressure in *B* being different from that in *A*, the position of the meniscus will accordingly alter, but by raising the one cup relative to the other so that the level of the fluid in the one is higher than that in the other, this extra pressure may be compensated for, until the bubble returns to its original position. The amount by which the one cup requires to be raised above the other will clearly give a measure of the difference of pressures in *A* and *B*. This raising or tilting is effected and measured by means of the horizontal wheel *W* and micrometer screw *S*. The amount of tilt, and consequently the difference in pressure, is measured on the scale at *T*. This type of gauge on which pressures down to 1/1000 in. of water may be measured was originally due to Prof. Chattock, and is in general use at the National Physical Laboratory.

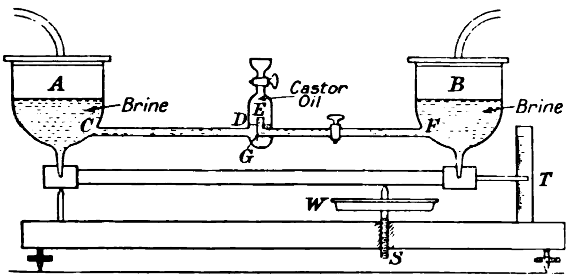


FIG. 20

§ **14.** It is by means of this type of gauge and the pitot tube already described that the calibration of the wind channels used for aerodynamical work is effected. The wind speed is determined by means of the pitot, and at the same time for each speed the corresponding pressure at a small hole in the side of the channel is measured by transmission to a pressure gauge. Once this calibration has been effected it is evidently possible at any future time to determine the speed of the wind in the channel from a measurement of the pressure at the side, and this is the method always adopted.

§ **15.** For most well-shaped bodies in the aerodynamic sense it may be stated that the resistance varies as the square of the speed. Theoretical considerations indicate, and experience has

resistance increases very rapidly and extra energy, ...at expended in overcoming the viscous forces, is ... the formation of sound waves. The shell is riding ...f the compression waves it produces. Beyond this ... the projectile penetrates into the medium more ... the waves it forms, the resistance coefficient once ... down to a constant value, in magnitude however ... greater than previously. Fig. 22 illustrates a ... that has been obtained by Prof. C. V. Boys, showing sound waves produced by the motion of a bullet.

The exact nature of the stream lines, the rate at which ...e being produced and the rate at which energy is being ...d in the form of sound waves, if such occur, all contribute ... the production of what is ultimately experienced as ...ce. It follows that in general there exists a unique relation ...n the magnitude of this resistance and the nature of the ... It is equally clear that if a body be moving in a viscous ...ith velocity v the nature of the flow and the stream lines ... it originates will not necessarily be identical with that ...ring when the same body is passing through the velocity ...ler an acceleration. The fluid which it leaves behind in the ...r case will be disturbed in a totally different manner from that ...he former. It might be anticipated in the light of these ...arks that the resistance of an accelerated body whose velocity ...any instant is v will differ from that of the same body when ...ving constantly with velocity v. It has, however, been found ...perimentally that the magnitude of the difference is of the order ... the weight of the fluid displaced by the body and consequently ...r th... ...h of aeronautics which deals with aeroplanes, the ...esis... ...e taken as independent of the acceleration and ...m... ...n of the velocity. In the case of larger bodies, ... airships, where evidently the weight of the mass ... of the same order as the total weight of the air-... ...e atta... ...derable importance and is usually ...sumi... ...he mass of the airship receives ...rease...

...t ... ...athematical analysis it is im-... ...out ... ...erimentally the various stages in

the flow past such comparatively simple shaped bodies as struts and aerofoils. This being so it will readily be appreciated, that a clear investigation of the case of a rotating propeller moving forward through the fluid will practically be beyond our powers at the present stage, both experimentally and theoretically. Viewed very generally it will be possible to state that the air in front will be sucked in and discharged behind in a converging stream within which the motion will be a complicated combination of translational, rotational and eddying energy. As far as possible a more exact investigation of this extremely intricate problem will be reserved for Chapter IX.

§ **18.** *Atmospheric Conditions.* It has been tacitly understood that the fluid with whose properties we have been concerned has been continuously preserved at constant temperature. It requires very slight consideration to see that had this condition been violated to any appreciable extent, the nature of the resultant flow would have been considerably modified. From Charles' and Gay-Lussac's law which states that $PV = RT$, P, V and T being the pressure, volume and temperature absolute of any portion of the fluid, and R a constant for constant mass, a variation in T in general affects the pressure and thus disturbs the motion of the fluid.

Under atmospheric conditions the temperature is never uniform but falls off rapidly from the surface of the earth upwards. Each part of the earth is more or less continually being heated up daily for many hours under the effect of the sun's rays and during the remaining time is screened. As a consequence of this diurnal variation in temperature a variation in pressure is set up in the immediate belt surrounding the earth and convection of air tends to take place from regions of higher temperature to those of lower. This general effect is however considerably modified by local conditions. The form of contour of the earth's surface obstructs the flow and originates eddying motion of various forms.

Extremely localised changes in temperature due among other things to a sheltering of certain regions from the sun's rays by clouds likewise bring into being convection currents. The rotation of the earth itself further complicates these comparatively simple conditions. The air which would otherwise flow directly from a point of high temperature to one of low becomes deflected by the

spin of the earth, so that instead of moving directly it circulates round the position of lower pressure, giving rise to so-called cyclonic conditions, or when the circulation is about regions of high pressure, so-called anti-cyclonic weather. For many aeronautical purposes of primary importance, both experimental and military, an accurate study of weather conditions, the general laws governing them and the particular and local conditions affecting them is essential. Air raids for an example, and this is especially important with airships, can only be executed when it is known generally that the prevailing type of weather will be favourable. Variations of temperature and pressure with height interfere with the efficient working of the engine. Winds and gusts, both horizontal and directed partially or wholly upwards or downwards, materially modify the aerodynamic performance of machines, and without a clear understanding of the origin of these disturbances reliable experimental work and accurate tests on complete aeroplanes cannot be carried out.

The general variation of temperature, pressure and density with height above the earth's surface is given in the following tables, in which is included also the dependence of viscosity on temperature.

Values of ρ and ν.

M in slugs, L in feet, T in seconds. Slug = mass of g lbs. wt.

		Units
ρ air	= 0·00237 at 15° C. and 760 mm. Hg	M/L^3
ρ water	= 1·94 or 815 times that of air	M/L^3
ν air	= 0·000144 at 0° C. and 760 mm. Hg	L^2/T
	= 0·000159 at 15° C.	
	= 0·000251 at 100° C.	
ν water	= 0·0000164 at 5°	L^2/T
	= 0·0000159 „ 6°	
	= 0·0000154 „ 7°	
	= 0·0000150 „ 8°	
	= 0·0000146 „ 9°	
	= 0·0000141 „ 10°	
	= 0·0000137 „ 11°	
	= 0·0000134 „ 12°	
	= 0·0000123 „ 15°	
	= 0·0000108 „ 20°	

Variation of Pressure, Temperature, and Density of the Atmosphere with Height.

Height ft.	Pressure Ins. Hg	Temp. Cent. Abs.	Density $\frac{1}{g}\cdot\frac{\text{lbs.}}{(\text{ft.})^3}$	Density Ratio	Density $\frac{1}{g}\cdot\frac{\text{lbs.}}{(\text{ft.})^3}$	Density Ratio	Temp. Cent. Abs.	Pressure Ins. Hg
0	30·0	282	0·00246	1·000	0·00237	1·000	279	29·00
2,000	28·1	280	0·00232	0·940	0·00224	0·940	277	26·90
4,000	26·2	278	0·00215	0·875	0·00208	0·880	274	24·95
6,000	24·4	277	0·00201	0·815	0·00196	0·825	271	23·15
8,000	22·65	275	0·00187	0·760	0·00185	0·780	267	21·35
10,000	20·9	272	0·00175	0·710	0·00172	0·725	263	19·75
12,000	19·35	269	0·00163	0·665	0·00163	0·685	259	18·20
14,000	17·9	266	0·00153	0·625	0·00151	0·635	253	16·75
16,000	16·55	263	0·00144	0·585	0·00142	0·600	249	15·45
18,000	15·25	258	0·00133	0·540	0·00134	0·565	244	14·25
20,000	14·05	254	0·00125	0·510	0·00126	0·530	240	13·10
22,000	12·95	249	0·00118	0·480	0·00118	0·495	235	11·95
24,000	12·00	245	0·00108	0·440	0·00109	0·460	230	10·95

CHAPTER III

MATHEMATICAL THEORY OF FLUID MOTION

INTRODUCTION

§ 1. In the previous chapter the nature of the flow of a viscous fluid under various conditions was discussed qualitatively from comparatively elementary considerations, and incidentally the distinction between a viscous fluid and the perfect fluid usually dealt with in mathematical theory was clearly indicated*. In the latter case it was explained that when the fluid was under the action of tangential forces not merely could it not withstand them permanently, as indeed is the essence of any fluid, but no internal forces of frictional nature are brought into play to resist the change. In actual reality there can be no such thing as a perfect fluid according to this conception, yet the theory of its motion that has been developed, and the laws that govern it, frequently serve as a useful guide in the theory of aeronautics. On account of the difficulties inherent in a complete quantitative discussion of the behaviour of a viscous fluid, very little progress has been made by mathematical physicists in that direction, but on the other hand, mathematical treatment of the perfect fluid has yielded considerably to analysis. The nature of the flow will in general differ widely from that of an actual fluid, but under certain circumstances, especially during so-called "stream line" flow, a certain amount of useful information is forthcoming.

§ 2. *Two-dimensional Flow.* Fortunately for the analyst most of the problems relating to fluid motion arising in aeronautical investigations can be considerably simplified by an initial assumption that does not in itself vitiate the results to any extent. Struts and aerofoils are in general so long in comparison

* § 12, Chap. II.

to their cross-section that the fact that they possess ends at all does not appreciably affect the nature of the motion of the fluid past them except in the immediate neighbourhood of these ends. In such an event, if the body be of uniform cross-section, fluid which before entering the region of the strut or aerofoil flowed in the plane XOY, perpendicular to the length, will not be deflected out of that plane, and the motion will remain two-dimensional. It will be seen that the generality of the conclusions arrived at will scarcely be affected at all by this assumption, and at the same time a considerable saving in mathematical labour will evidently be involved.

A large amount of experimental evidence has been accumulated during the progress of aeronautical research indicating that up to speeds approaching the velocity of sound, the pressures brought into existence by motion through the air are such as would arise were that medium acting as though it were incompressible. It is

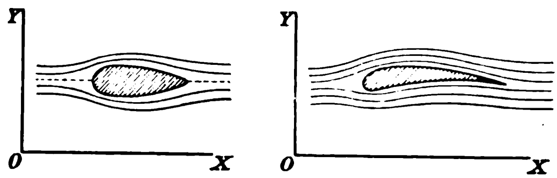


FIG. 23

not until energy is dissipated in the form of sound waves that the elasticity of the air affects the question to any degree, and since normal aeronautical speeds do not approach anywhere within that critical region there can be no error involved in assuming that the density of the fluid to be discussed in this chapter remains constant. It is only in the supposition that viscous forces do not exist that conclusions derived for the ideal fluid can fail in their application to reality.

§ 3. *Derivation of the Equations of Motion.* The equations determining the motion must be derived by a mathematical statement of the two basic facts of the nature of the fluid:

(1) The mass of the fluid within any region remains constant for all time, since, the medium being incompressible, there can be no accumulation of fluid there.

(2) The motion of every fluid element is consistent with Newton's Laws of Motion.

In a sense these two statements and all that they involve are a definition of the nature of the fluid and all deductions regarding its behaviour can only be a recasting of these into a new but equivalent form. Consider a small rectangular slab of the fluid $ABCD$ such that $DC = \delta x$ and $CB = \delta y$, and suppose u and v to be the velocity components horizontally and vertically at the mid-points of DA and DC respectively, then the corresponding components at the mid-points of AB and CB are $v + \frac{\partial v}{\partial y}\delta y$ and $u + \frac{\partial u}{\partial x}\delta x$ respectively, since u and v are increasing in their corresponding directions at rates $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial y}$.

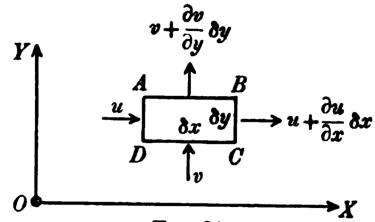


FIG. 24

The quantity of fluid entering the small element $ABCD$ in the horizontal direction is accordingly $\rho u\,\delta y$ and that leaving $\rho\left(u + \frac{\partial u}{\partial x}\delta x\right)\delta y$, where ρ is the density of the fluid.

In the same way the amount entering in the y direction is $\rho v\,\delta x$ and that leaving $\rho\left(v + \frac{\partial v}{\partial y}\delta y\right)\delta x$.

Statement (1) demands that the total quantity of fluid within $ABCD$ must not increase, and accordingly it follows that

$$\rho u\,\delta y - \rho\left(u + \frac{\partial u}{\partial x}\delta x\right)\delta y + \rho v\,\delta x - \rho\left(v + \frac{\partial v}{\partial y}\delta y\right)\delta x = 0,$$

and hence $$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \text{.......................(1).}$$

This is the equation of continuity of mass and is the mathematical equivalent to statement (1).

For an interpretation of statement (2) the forces which are acting upon the small element $ABCD$ producing the motion must be analysed.

For convenience these will be considered as of two kinds.

(*a*) External forces such as gravity for example.

Let X and Y be the values of the force components per unit mass, then $\rho X\delta x\delta y$ and $\rho Y\delta x\delta y$ are the component forces acting on the elementary area $ABCD$.

(*b*) The pressure due to the surrounding fluid on the element *ABCD*.

If p be the pressure at the mid-point DA, then

$$p + \frac{\partial p}{\partial x}\delta x$$

will be the back pressure at the mid-point of CB and the total resultant thrust in the direction OX on the element then becomes

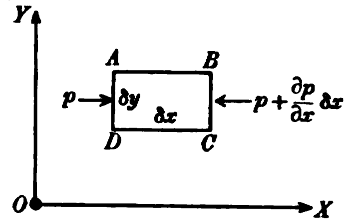


FIG. 25

$$p\delta y - \left(p + \frac{\partial p}{\partial x}\delta x\right)\delta y = -\frac{\partial p}{\partial x}\delta x\delta y.$$

Newton's Second Law applied to the element in the direction OX then takes the form

$$\rho X \delta x \delta y - \frac{\partial p}{\partial x}\delta x \delta y$$

$$= \rho\, \delta x\, \delta y \times (\text{acceleration of element } ABCD \text{ in direction } OX).$$

Now the velocity at any point in the fluid will depend not merely on the geometrical coordinates of the point but also on time. Even at a fixed point in the medium the velocity will vary from second to second and therefore u and v are functions of x, y and t.

Hence

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial t} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{\partial u}{\partial t},$$

and this is the expression for the acceleration in the horizontal direction. The equation of motion thus becomes

$$\rho X \delta x \delta y - \frac{\partial p}{\partial x}\delta x \delta y = \rho \delta x \delta y\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right)$$

or
$$X - \frac{1}{\rho}\frac{\partial p}{\partial x} = \left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) \qquad \text{...............(2).}$$

By exactly the same process from a consideration of the forces in the direction OY we obtain

$$Y - \frac{1}{\rho}\frac{\partial p}{\partial y} = \left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) \qquad \text{...............(3).}$$

Equations (1), (2) and (3) are between them a mathematical expression of the statements (1) and (2) previously discussed. As

such they represent the fluid completely in the mathematical sense, and from them all the properties of the motion are to be derived. They are three partial differential equations in u, v and p and they indicate that these three quantities, and therefore the motion, are completely determinate from a knowledge of X, Y, the initial conditions of motion and the permanent boundaries. The actual expressions for the integrals of the equations would involve arbitrary functions and constants since the derivatives entering into the equations are partial, and it is these arbitrary functions and constants which are determined by the initial conditions of motion and a knowledge of the permanent boundaries.

§ 4. It is proposed at this stage to restrict the motion in two distinct ways. In the first place it will be assumed that no external forces X and Y are operating; that in fact pressures alone are the only forces responsible for the motion. That this will not in itself destroy the validity of an application of results to actual problems of a certain class is evident from the fact that gravity does not determine to any extent the motion of air past a body moving through it. In the second place it will be supposed that the motion is steady, that the velocity components and pressures do not vary with time. This involves that at every point in the fluid fixed relative to the axes the speed and direction of motion never vary. It will accordingly now be possible to simplify the equations derived by putting

$$X = Y = 0, \quad \frac{\partial u}{\partial t} = \frac{\partial v}{\partial t} = 0,$$

and the fundamental equations of motion take the form

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \text{.................................(4)},$$

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} \quad \text{....................(5)},$$

$$-\frac{1}{\rho}\frac{\partial p}{\partial y} = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} \quad \text{.................(6)}.$$

As they stand these equations are not very suitable for solution but they can be rapidly recast into a much more suggestive form.

It will be found in text books on the Differential Calculus, and it can be easily verified, that (4) is simply the condition that a function ψ exists such that

$$u = -\frac{\partial\psi}{\partial y}, \quad v = \frac{\partial\psi}{\partial x} \quad \text{.....................(7).}$$

In essenee the two equations (7) are identical with (4) and no limitation is thereby involved on the nature of the problem.

§ 5. Various aspects of the question are readily represented in terms of this new function.

If q be the resultant velocity at any point in the fluid, then

$$q^2 = u^2 + v^2 = \left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial y}\right)^2 \quad \text{..............(8).}$$

Since u and v are the component velocities, v/u is the slope of the direction of motion at the point, but

$$v/u = -\frac{\partial\psi}{\partial x}\bigg/\frac{\partial\psi}{\partial y}$$

= slope of the curve determined by $\psi =$ constant. That this is so can be easily seen by differentiating ψ; thus

$$\frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial y}\frac{dy}{dx} = 0,$$

$$\therefore \frac{dy}{dx} = -\frac{\partial\psi}{\partial x}\bigg/\frac{\partial\psi}{\partial y}.$$

The remarkable result follows at once that the series of curves represented by

$$\psi = \text{constant}$$

and obtained by giving a series of values to the constant are simply the stream lines in the problem, since the direction of motion coincides at every point with the direction of the curve of this system that passes through the point.

The problem of the motion of a perfect fluid now reduces itself to determining an expression ψ as a function of x and y satisfying the boundary and initial conditions, the component velocities and pressures being given by (5), (6) and (7). On the other hand the reverse process may be adopted, by choosing some function arbitrarily and determining the pressures and boundary conditions required to maintain the motion.

By means of this new function the pressure equations (5) and (6) can after a little manipulation be thrown into a much more convenient form, viz.

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{1}{2}\frac{\partial}{\partial x}(u^2 + v^2) + v\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right)$$

$$= \frac{1}{2}\frac{\partial}{\partial x}(q^2) - \frac{\partial \psi}{\partial x}\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right)$$

$$-\frac{1}{\rho}\frac{\partial p}{\partial y} = \frac{1}{2}\frac{\partial}{\partial y}(q^2) - \frac{\partial \psi}{\partial y}\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right).$$

Multiplying these equations by dx and dy respectively and adding, remembering that

$$dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy,$$

$$d(q^2) = \frac{\partial (q^2)}{\partial x}dx + \frac{\partial (q^2)}{\partial y}dy,$$

$$d\psi = \frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy,$$

we obtain on integrating

$$-\frac{p}{\rho} = \frac{q^2}{2} - \int d\psi\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right) + \text{constant} \quad \text{.........(9)}.$$

If p_0 be the pressure on the stream line ψ_0, where the velocity is q_0, then

$$\frac{p - p_0}{\rho} = \frac{q_0^2 - q^2}{2} + \int_{\psi_0}^{\psi} d\psi \nabla^2 \psi \quad \text{............(10)},$$

where $$\nabla^2 \psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \quad \text{..........................(11)}.$$

Equation (10) thus provides a means of determining the pressure at any point in terms of the velocity at that point and the function ψ for the system of stream lines when the motion is steady and there are no external forces. If the motion be unsteady the student can easily verify that an arbitrary function of t must be added to the right-hand side. The initial and boundary conditions are given by p_0, q_0 and ψ_0. It need scarcely be pointed out that any stream line may be taken as one of the fixed boundaries.

§ 6. It is clear that complications may arise in the evaluation of the integral in (9) except when $\nabla^2\psi$ can be expressed explicitly in terms of ψ. It will be convenient to separate the problems that arise into two classes, viz. those in which

$$(a)\quad \nabla^2\psi = 0$$

$$\text{and } (b)\quad \nabla^2\psi = f(x, y).$$

To understand exactly the true nature of this classification it will be necessary to enquire closely into the significance of the expression $\nabla^2\psi$. Remembering that

$$u = -\frac{\partial\psi}{\partial y}, \text{ and } v = \frac{\partial\psi}{\partial x},$$

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = +\frac{\partial}{\partial x}\left(+\frac{\partial\psi}{\partial x}\right) - \frac{\partial}{\partial y}\left(-\frac{\partial\psi}{\partial y}\right)$$

$$= \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}.$$

Now the most general displacement of an element of a fluid consists of a pure strain compounded with a rotation of angular velocity

$$\frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right).$$

This is also well known in the elementary theory of elasticity. The value that $\frac{1}{2}\nabla^2\psi$ assumes at any point of the fluid then represents the rotational velocity of the element of the fluid situated there at the moment. It is frequently termed the *spin* or *rotation* at the point. The significance may, however, be brought out from another point of view.

§ 7. *Flow and Circulation.* If A and B be any two points in a fluid the value of the integral

$$\int_A^B (udx + vdy)$$

taken along a path joining A to B is called the *flow* along that path from A to B. When the curve is closed so that A and B coincide the value of the integral is called the *circulation*.

Applying this to a small rectangular element of fluid $ABCD$ of area $\delta x\, \delta y$,

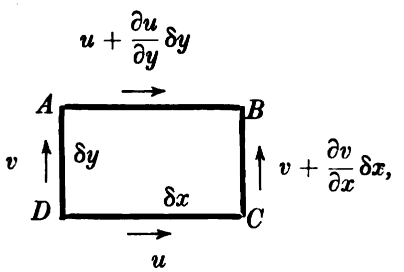


FIG. 26

the circulation round $ABCD$ is

$$u\,\delta x + \left(v + \frac{\partial v}{\partial x}\delta x\right)\delta y - \left(u + \frac{\partial u}{\partial y}\delta y\right)\delta x - v\,\delta y$$

$$= \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\delta x\,\delta y = \tfrac{1}{2}\nabla^2\psi\,\delta x\,\delta y.$$

From this point of view $\frac{1}{2}\nabla^2\psi$ represents the circulation per unit area or strength of circulation at the point in the fluid, while from the earlier interpretation it appears as the intensity of spin or rotation. Generally speaking therefore, the classification of the problems that arise amounts simply to a separation into cases in which there is no spin, or, as it will be referred to here, in which there is irrotational motion of the fluid, and those in which spin does exist and is distributed over the region of the fluid in a manner determined by the expression $f(x, y)$.

§ 8. *Irrotational Motion.*

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(12).$$

Rewriting this in the form

$$\frac{\partial}{\partial x}\left(\frac{\partial\psi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial\psi}{\partial y}\right) = 0,$$

and applying the same argument as was advanced when introducing the stream function, this equation is simply the condition that $\frac{\partial\psi}{\partial x}$ and $\frac{\partial\psi}{\partial y}$ are related through a new function ϕ such that

$$\frac{\partial\psi}{\partial x} = -\frac{\partial\phi}{\partial y} \text{ and } \frac{\partial\psi}{\partial y} = \frac{\partial\phi}{\partial x} \quad\ldots\ldots\ldots\ldots\ldots(13).$$

In the case of irrotational motion u and v can accordingly be expressed in terms of a new function ϕ such that

$$u = -\frac{\partial \psi}{\partial y} = -\frac{\partial \phi}{\partial x}, \quad v = \frac{\partial \psi}{\partial x} = -\frac{\partial \phi}{\partial y},$$

and
$$q^2 = \left(\frac{\partial \psi}{\partial x}\right)^2 + \left(\frac{\partial \psi}{\partial y}\right)^2 = \left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2.$$

Moreover ϕ satisfies the equation (12) since

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial x \partial y} = 0 \text{..............(14)}.$$

The geometrical interpretation of this new function and its relation to the stream function is at once evident from the fact that

$$\frac{v}{u} = \frac{\frac{\partial \phi}{\partial y}}{\frac{\partial \phi}{\partial x}},$$

which implies that the curves determined by $\phi =$ constant, usually known as the equipotential lines, are the orthogonal trajectories of the stream lines determined by $\psi =$ constant. Since these two expressions moreover satisfy the same differential equations (12) and (14) they are interchangeable as far as expressions for stream function of possible types of irrotational motion are concerned. The system of stream lines and equipotentials divide the region of the fluid into a system of infinitely small rectangles.

§ **9.** *Discussion of the equation*

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \text{.......................(12)}.$$

The real key to the solution of any problem in the irrotational motion of a non-viscous fluid lies in the determination of the appropriate expression that satisfies this equation and at the same time gives the requisite boundaries to the fluid. The pressure will then be given by the modified form of (10), viz.

$$\frac{p - p_0}{\rho} = \frac{q_0^2 - q^2}{2} \text{.....................(15)}.$$

Equation (12) is of a well-known type and its solution is easily verified to be given by equating ψ to the real or to the imaginary

part of $f(x+iy)$, where $i^2 = -1$. In fact, since ϕ and ψ must both satisfy this equation, we may write

$$\phi + i\psi = f(x+iy) \qquad \text{.....................(16).}$$

That the equations (12), (13) and (14) are immediately derivable from this becomes clear on differentiating, for

$$\frac{\partial\phi}{\partial x} + i\frac{\partial\psi}{\partial x} = f'(x+iy),$$

$$\frac{\partial\phi}{\partial y} + i\frac{\partial\psi}{\partial y} = if'(x+iy) = i\left(\frac{\partial\phi}{\partial x} + i\frac{\partial\psi}{\partial x}\right) = i\frac{\partial\phi}{\partial x} - \frac{\partial\psi}{\partial x},$$

and equating real and imaginary parts equations (13) are shown to hold. From these alone (12) and (14) both follow. It would appear then that any number of problems of two-dimensional irrotational flow and their solutions may be constructed by simply choosing functions of the complex variable $z = x + iy$ and equating the real and imaginary parts to the potential and stream functions ϕ and ψ respectively.

On the other hand, the determination of the particular function of this nature to satisfy certain presupposed boundary conditions is comparatively difficult to effect, and the method of solution has frequently been the inverse one indicated above.

§ **10.** Consider in illustration the case

$$\phi + i\psi = z + 1/z = x + iy + \frac{1}{x+iy}$$

$$= x + iy + \frac{x-iy}{x^2+y^2} = \frac{x(x^2+y^2+1)}{x^2+y^2} + i\frac{y(x^2+y^2-1)}{x^2+y^2}.$$

The stream lines are then given by

$$\psi = y(x^2+y^2-1)/(x^2+y^2).$$

The particular stream line $\psi = 0$ corresponds to

$$y = 0 \text{ and } x^2+y^2-1=0,$$

i.e. the X-axis and the circle of radius unity may be taken as fixed boundaries. The other stream lines are determined by substituting a series of constant values for ψ. The final diagram of the flow is given in fig. 27 where it is seen that the motion is that of a circular cylinder of unit radius moving with constant speed in a perfect irrotational fluid.

The velocity at any point is obtained by remembering that

$$u = -\frac{\partial\psi}{\partial y} = -1 + \frac{x^2-y^2}{(x^2+y^2)^2}, \quad v = \frac{\partial\psi}{\partial x} = \frac{2xy}{(x^2+y^2)^2}.$$

Hence at an infinite distance $u = -1$, $v = 0$, and on the surface of the body at $x = 1, y = 0, u = 0, v = 0$, so that the circular cylinder is stationary and the fluid is streaming past it with unit velocity at infinity.

§ 11. The curves corresponding to $\phi =$ constant, the orthogonal trajectories to the stream lines, are indicated in the figure by dotted lines and the systems $\phi =$ constant and $\psi =$ constant map out the field into an infinite number of infinitely small rectangles. Each point in the fluid lying as it does at the point of intersection of a curve of the one system with one of the other will be associated with a particular value of ϕ and ψ defining it in general uniquely. We may in fact construct a new plane w, of axes ϕ and ψ, and each point in it will correspond to a point in the fluid in the z-plane. Each infinitely small rectangle in the z-plane bounded by neighbouring stream lines and equipotential

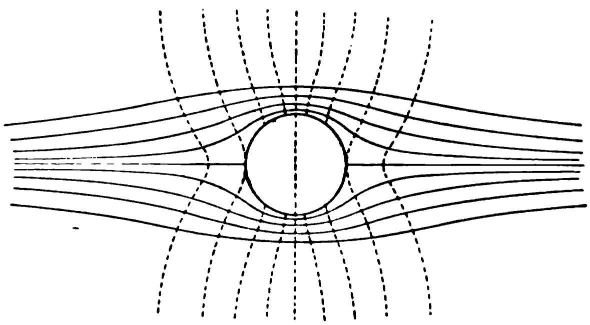


FIG. 27

lines will correspond to an infinitely small rectangle in the w-plane bounded by lines parallel to the ϕ and ψ axes along which these functions are constant. The relative dimensions of these corresponding rectangles will depend of course on the exact relationship between z and w, and a knowledge of these dimensions for the whole region of the fluid would be equivalent to a solution of the problem. It can be shown however that every functional relationship of the type here known to exist, viz. $w = f(z)$, demands that such corresponding rectangles are always similar. The region of the fluid in the z-plane is under these circumstances said to be conformally transformed upon the region in the w-plane.

§ **12.** Normally it is no simple matter to determine from a given problem what is the requisite formula of transformation

giving such a correspondence between the z- and the w-planes, but the process is frequently facilitated by an intermediate step. An attempt is made to transform the z-plane and the w-plane conformally on the upper half of a new t-plane so that both z and w are expressed in terms of a new variable t, and these together provide the functional relation between z and w required.

The fundamental theorem upon which most of these transformations depends is due to Schwarz and Christoffel, who succeeded in establishing the formula for transforming the region bounded by a polygon conformally upon the upper half of another plane. A large class of problems it will be seen ultimately depend for their solution upon this step. It must be noticed that so long as complex functions are dealt with, the hydrodynamical equations will be satisfied and it will only be necessary therefore to consider the boundaries. If a functional relation existing between two planes is such as to provide a correspondence between the boundaries in these planes it is the transformation required.

Consider the relation between the two complex variables z and t,

$$\frac{dz}{dt} = A\,(t - t_1)^{\frac{\alpha}{\pi}-1}\,(t - t_2)^{\frac{\beta}{\pi}-1}\ldots \qquad \ldots\ldots\ldots(17).$$

where $t_1, t_2, \ldots$ are real quantities ranging from $-\infty$ to $+\infty$ and $\alpha, \beta, \ldots$ the internal angles of a polygon so that

$$\alpha + \beta + \ldots = (n - 2)\,\pi.$$

Consider the variation in the complex quantity

$$\delta z = A\delta t\,(t - t_1)^{\frac{\alpha}{\pi}-1}\,(t - t_2)^{\frac{\beta}{\pi}-1}\ldots$$

as t, always remaining real, moves cautiously along the real axis in the t-plane from $+\infty$ to $-\infty$. A may be either a real or complex constant.

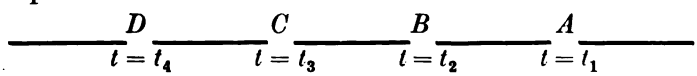


FIG. 28

As long as t lies to the right of the point A all the variable factors in the expression for δz will remain real. This implies that the modulus only of δz for this range will change in value while its amplitude, i.e. its inclination, will remain fixed. z will consequently move along a straight line.

After passing through $t = t_1$, $t - t_1$ becomes negative, and as

long as t lies between A and B, all the other variable factors will remain unchanged in sign and real. The term

$$(t-t_1)^{\frac{a}{\pi}-1}=(t_1-t)^{\frac{a}{\pi}-1}(-1)^{\frac{a}{\pi}-1}.$$

The first portion of this is now real and it is seen that the effect of passing through $t=t_1$ has been to introduce into the expression for δz a factor

$$\begin{aligned}(-1)^{\frac{a}{\pi}-1}&=(\cos\pi-i\sin\pi)^{\frac{a}{\pi}-1}\\&=\cos(\pi-\alpha)+i\sin(\pi-\alpha).\end{aligned}$$

Since the amplitude of the product of two complex numbers is the sum of the amplitudes of the two constituents, this means that the amplitude of δz after passing through the point A has increased by $\pi-\alpha$ and from then onwards maintains a constant amplitude until it arrives at the point B. The corresponding figure so far in the z-plane, fig. 29, clearly consists of the two sides of a polygon inclined at an angle $\pi-\alpha$. A similar effect is produced in passing through B, C, etc., and it becomes clear that the formula transforms the real axis in the t-plane into the sides of a polygon in the z-plane of internal angles α, β, etc. The expression for z derived by integration then provides the formula transforming the region bounded by this polygon conformally on the upper half of the t-plane where t is now supposed complex.

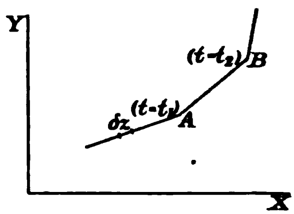


FIG. 29

The foregoing verification of this theorem must not be taken as a proof, and for a mathematically sound treatment the student is referred to a text book on the Theory of Functions.

§ **13.** As a simple application of the method thus suggested consider the problem where fluid, flowing horizontally at infinity with velocity V, is supposed to impinge on a flat plate situated perpendicularly to the general direction of the stream (fig. 30).

Let CC' be the plate and suppose $ABCDA'$ to be the stream line $\psi=0$ which meets the plate directly at B. Since the flow is symmetrical about AA' only the upper half of the z-plane need be considered.

Consider the relation between z and t that transforms this region of the fluid bounded by the infinite polygon $ABCDA'$ conformally to the upper half of a t-plane, where

$$\begin{array}{lll} A \text{ corresponds to} & t = \infty, \\ B \quad ,, & t = 1, \\ C \quad ,, & t = 0, \\ D \quad ,, & t = -1, \\ A' \quad ,, & t = -\infty. \end{array}$$

The internal angles of this polygon are

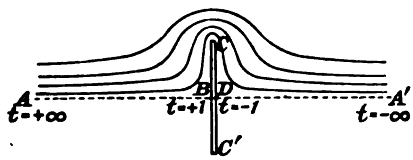
$$\frac{\pi}{2} \text{ at } t = 1 \text{ and } t = -1,$$

$$2\pi \text{ at } t = 0.$$

Hence from the Schwarz-Christoffel Theorem

$$\frac{dz}{dt} = A(t-1)^{-\frac{1}{2}}\, t\,(t+1)^{-\frac{1}{2}} = At/\sqrt{(t^2-1)}.$$

$$\therefore z = A\sqrt{(t^2-1)} + B.$$



z-plane.

FIG. 30

Let the origin in the z-plane be taken at B, so that for $t = \pm 1$, $z = 0$ and for $t = 0, z = li$, where $2l$ is the total breadth of the plane, then $B = 0$ and $A = l$.

$$\therefore z = l\sqrt{(t^2-1)}.$$

For the construction of the w-plane we notice that the fluid is bounded by $\psi = 0$ and a stream line infinitely far away, i.e. $\psi = \infty$, so that the corresponding region is thus the upper half of the w-plane. This is clearly transformed to the corresponding upper half of the t-plane by

$$w = Ct + D.$$

If the point C of the plate be taken as $\phi = 0$, $\psi = 0$, then $w = 0$ when $t = 0$,

$$\therefore w = Ct,$$

$$z = l\sqrt{(t^2-1)}.$$

These two equations are the solution of the hydrodynamical problem provided the constant C be determined, and this must evidently depend on the conditions of the fluid at an infinite distance where the fluid is supposed flowing steadily with velocity V horizontally. Now

$$\text{Velocity} = \frac{dw}{dz} = \frac{dw}{dt} \cdot \frac{dt}{dz} = C\frac{\sqrt{(t^2-1)}}{lt}.$$

When $|t| \to \infty$ this limits to C/l,

$$\therefore C = lV,$$

and accordingly $\quad w = lVt, \quad z = l\sqrt{(t^2-1)},$

or $\quad z^2 = l^2(t^2-1) = (w^2 - l^2V^2)/V^2,$

or $\quad w^2 = V^2(z^2 + l^2).$

A separation of the real and the imaginary terms then provides as the expression for the system of stream lines

$$\psi^4 + V^2(x^2 - y^2 + l^2)\psi^2 - V^2x^2y^2 = 0.$$

It may be remarked that from the symmetry of the solution with regard to x and y the corresponding pressures on each side of the plate are equal and oppositely directed and consequently no resistance will be experienced. An examination of the velocity at various points on the plate, viz.

$$\frac{dw}{dz} = V^2z/w = Vz/\sqrt{(z^2 + l^2)},$$

shows that it is zero at $z = 0$, i.e. the centre of the plate, and infinite at $z = \pm li$, i.e. at the edges C and C'.

This illustration may be taken as typical of the species of problem that may be solved by the direct application of the Schwarz-Christoffel transformation to the z- and w-planes, but it is evident that its success will be seriously limited by the fact that the boundaries must be plane sides of a polygon. There is, however, an extension of the theorem to curved boundaries for which the student must be referred to the original papers*. Considerably greater freedom may be obtained if it be remarked that the application need not be restricted to the z-plane but to any other plane that may be derived from it or from any combination of it and w. If for example the variation of $\dfrac{dz}{dw}$

* "Applications of Conformal Representation to Hydrodynamics," by J. G. Leathem, *Trans. Roy. Soc.* 1915. "Discontinuous Motion past a Curved Boundary," by H. Levy, *Proc. Roy. Soc.* 1916.

over the boundaries is known for any problem and if when represented on a $\frac{dz}{dw}$-plane the new boundaries are those of a polygon, the method may be applied to this and the final relation between $\frac{dz}{dw}$ and w derived. One further integration will then express the required function of z and w. A number of very interesting cases solved by this method will be shortly forthcoming when "discontinuous" motion is treated.

§ **14.** A particular case of the flow past a flat plate, where both l and V are taken as unity, provides the expression

$$z = \sqrt{(w^2 - 1)}.$$

In a previous example the flow past a circular cylinder provided as solution

$$z = w + \sqrt{(w^2 - 1)},$$

differing from the above by the presence of the linear term w. Now $z = w$ represents the problem of steady streaming with unit velocity parallel to the horizontal axis, and this suggests at once that a series of cases of flow past obstacles may be derived by the superposition in varying degree of a number of solutions of known problems. It will be clear that this indirect method of attack does not furnish a method of obtaining the solution of any proposed problem but rather furnishes the solution from which the problem is obtained. Nevertheless, it will become apparent that the method admits of a considerable amount of latitude and generalisation, and will thus allow to a certain degree a choice of problem.

Example 1. Superpose the steady streaming $z = Aw$ upon the steady flow past a flat plate $z = B\sqrt{(w^2 - 1)}$,

$$z = Aw + B\sqrt{(w^2 - 1)}.$$

$$\therefore \frac{dz}{dw} = A + Bw/\sqrt{(w^2 - 1)}.$$

When $\qquad |w| \to \infty, \quad \frac{1}{V} = \underset{|w| \to \infty}{\mathrm{L}} \frac{dz}{dw} = A + B,$

where V is the velocity at an infinite distance.

The stream line $\psi = 0$ is generally

$$x + iy = A\phi + B\sqrt{(\phi^2 - 1)}.$$

When $+\infty > \phi > 1$, $x = A\phi + B\sqrt{(\phi^2 - 1)}$ and $y = 0$.

When $$1 > \phi > -1,$$

$$x + iy = A\phi + iB\sqrt{(1-\phi^2)},$$

$$\therefore x = A\phi, \quad y = B\sqrt{(1-\phi^2)},$$

$$\text{i.e. } \frac{x^2}{A^2} + \frac{y^2}{B^2} = 1,$$

an ellipse of semi-axes A and B.

When $$-1 > \phi > -\infty,$$

$$x + iy = A\phi - B\sqrt{(\phi^2 - 1)},$$

$$\therefore x = A\phi - B\sqrt{(\phi^2-1)}, \quad y = 0.$$

The problem is thus that of the steady symmetrical flow of a fluid past any elliptic cylinder of the series $\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1$.

The student will find it an interesting exercise to discuss the more general case where A and B are not restricted to be real.

Example 2. Superpose two cases of the steady flow past a plate, e.g. $$z = A(w^2 - 1) + B(w^2 - 4).$$

As before the velocity at an infinite distance is $1/(A + B)$ and is horizontal, and the obstacle is the elliptic cylinder

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} = 3,$$

with a portion of the minor axis projecting beyond the surface of the cylinder.

RESISTANCE

§ **15.** An examination of the system of stream lines representing the flow past a symmetrical body like a cylinder shows that the pressures round the surface must balance, and consequently the fluid exerts no resultant thrust on the body. That this theorem is of general validity for the steady motion of any body in a perfect fluid may be shown in a variety of ways. The pressures, for example, on each element of the surface may be calculated, and the components integrated over the whole surface.

This absence of reaction between body and fluid is extremely unfortunate for it implies an essential failure in the application of results obtained for a perfect fluid to a real case. Mathematical physicists have striven for years to introduce some new assumption into the nature of the flow that will avoid this fatal result, but it

is clear that no matter how ingenious the suggestions may be, they must of necessity be artificial since they attempt to simulate the action of viscosity without actually assuming its existence.

Perhaps the most successful of these attempts is that which has given rise to the so-called "discontinuous motion" or "free surface" theory. It has been shown that the pressure at any point of a fluid in steady motion can be represented by

$$p - p_0 = \tfrac{1}{2}\rho\,(v_0^2 - v^2) \quad \ldots\ldots\ldots\ldots\ldots\ldots(15),$$

where p_0 and v_0 are the pressures and velocity at a distance at which the motion is undisturbed by the presence of the body. Now it is a necessary condition in any physical problem that a pressure at any point less than a perfect vacuum cannot arise, and it consequently follows from this equation that the velocity must nowhere increase beyond a certain maximum. It can, however, easily be shown that according to the type of flow so far considered the velocity can be made to reach any desirable magnitude by increasing the curvature of the body. At the sharp edge, for example, of a flat plate the velocity becomes infinite and consequently the pressures infinitely great and negative. This violation of a physical assumption has originated the suggestion that the fluid, fig. 31, instead of flowing round the sharp edge A of the body, breaks away into the line AC, dividing the region behind the body of the fluid at rest from the remainder in motion. That such a state of affairs does not actually occur in practice must be admitted, but under certain circumstances, especially in the immediate neighbourhood of the edges AB, an approximation to this does take place. The fluid in the region behind the body being at rest the pressure there is everywhere constant, and it follows from the pressure equation that the velocity along the free stream lines AC and BD as they are called is constant and equal to that of the fluid at infinity V. Whether or not this type of motion can arise in a perfect fluid will be discussed later, but for the present it will be assumed to exist and a determination of the stream lines and the resistance carried through.

§ **16.** Consider the case of the flat plate AB. From symmetry IO will be a stream line dividing at O where the velocity will be zero, and thence flowing in the directions $OAC\ldots$ and $OBD\ldots$.

The boundaries of the fluid in the z-plane are OAC and OBD,

where OA and OB are straight lines, but the shapes of AC and BD are for the present not known, and not necessarily straight lines, and it will consequently not be possible to apply the Schwarz-Christoffel theorem directly to effect a transformation of the region of the fluid in the z-plane to the upper half of a t-plane. One fact however is known about the stream line AC—the velocity along it is V. For one part of the boundary the *direction* of motion is constant, and for the other the *magnitude*. If therefore a new complex plane, transformable to the z-plane by some known relation, could be derived in which the one axis was a function of velocity and the other of direction of motion, the boundaries in it corresponding to those of the z-plane would be straight lines.

Such a relation can be simply derived as follows:

$$\frac{dw}{dz} = \frac{\partial\phi}{\partial x} + i\frac{\partial\psi}{\partial x} = u - iv \quad \text{...............(18)},$$

$$\therefore \log\frac{dz}{dw} = \log\frac{u+iv}{u^2+v^2} = \log\frac{1}{q}\left(\frac{u}{q} + i\frac{v}{q}\right) = \log\frac{1}{q}(\cos\theta + i\sin\theta)$$

$$= \log\frac{1}{q}\epsilon^{i\theta} = \log\frac{1}{q} + i\theta,$$

where q is the resultant velocity and θ the direction of motion, at any point. This formula provides a new and convenient plane of representation for the type of problem under discussion.

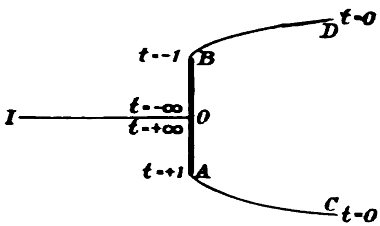


FIG. 31

It will be assumed that the values of t at the various points are as shown in the figure and that the velocity at an infinite distance is unity, so that this is also the constant velocity along the free stream line. Writing

$$\Omega = \log\frac{dz}{dw} = \log\frac{1}{q} + i\theta \quad \text{..................(19)},$$

along OA, $\quad \theta = -\pi/2$,

at O, $\quad q = 0, \quad \therefore \log 1/q = +\infty$,

at A, $\quad q = 1, \quad \therefore \log 1/q = 0$.

along AC, $\quad -\pi/2 \leqslant \theta \leqslant 0$.

$\quad q = 1, \quad \therefore \log 1/q = 0$.

along OB, $\quad \theta = +\pi/2$,

at B, $\qquad q = 1, \quad \therefore \log 1/q = 0,$

along BD, $\qquad 0 \leqslant \theta \leqslant + \pi/2.$

The following figure may accordingly be constructed for the Ω-plane
$$\Omega = \log 1/q + i\theta,$$

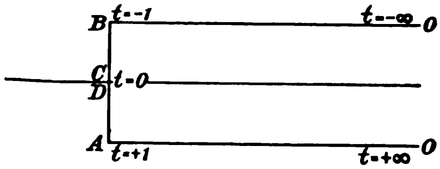


Fig 32

where the corresponding values of t are given, the polygon to be transformed by the Schwarz-Christoffel method being $OABO$ of internal angles $\pi/2$ both at A and B.

$$\text{Hence } \frac{d\Omega}{dt} = C\,(t-1)^{-1/2}\,(t+1)^{-1/2} = C/\sqrt{(t^2-1)},$$

and therefore
$$\Omega = C \log\,[t + \sqrt{(t^2 - 1)}] + D.$$

To determine the constants we note that

$$+\infty > t \geqslant 1, \qquad \log 1/q + i\theta = C\log\,[t + \sqrt{(t^2-1)}] + D.$$

$$\text{At } A, \qquad t = 1, \quad q = 1, \quad \theta = -\pi/2,$$
$$\therefore D = -\pi i/2.$$

$$1 \geqslant t \geqslant 0, \qquad \log 1/q + i\theta = C\log\,[t + i\sqrt{(1-t^2)}] + D$$
$$= Ci\tan^{-1}\sqrt{(1-t^2)}/t + D,$$

$$\log 1/q + i\theta = C\log\,[-t - i\sqrt{(1-t^2)}] + C\log(-1) + D$$
$$= C\log\,[-t - i\sqrt{(1-t^2)}] + C\pi i + D$$
$$= C\tan^{-1}\sqrt{(1-t^2)}/t + C\pi i + D.$$

$$-1 \geqslant t \geqslant -\infty, \quad \log 1/q + i\theta = C\log\,[-t + \sqrt{(t^2-1)}] + C\pi i + D.$$

$$\text{At } B, \qquad t = -1, \quad q = 1, \quad \theta = \pi/2,$$
$$\therefore C = 1.$$

$$\text{Hence} \quad \log\frac{dz}{dw} = \Omega = \log\,[t + \sqrt{(t^2-1)}] - i\pi/2,$$

$$\therefore \frac{dz}{dw} = -i[t + \sqrt{(t^2-1)}].$$

It remains to determine the appropriate relation between w and t.

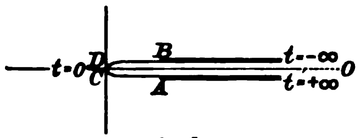
Suppose that at the point of divergence of the stream line

$\psi = +0, \psi = -0$ at 0, the value of ϕ is zero. The w-plane is then as shown in fig. 33.

w-plane.

FIG. 33

The region occupied by the fluid now corresponds to the whole of the w-plane bounded internally by the two lines OC and OD lying infinitely close to the axis. For a direct application of the Schwarz-Christoffel method it will be necessary to deal with the w^{-1}-plane where the point O at which t becomes infinite will be at infinity.



w^{-1}-plane.

FIG. 34

The internal angle of the polygon is 2π at $t = 0$,

$$\therefore \frac{dw^{-1}}{dt} = 2t/E,$$

$$\therefore w^{-1} = t^2/E + F.$$

When $t = 0$, $w^{-1} = 0$ since ϕ is infinite there, and consequently $F = 0$.

$$\therefore w = E/t^2$$

and
$$\frac{dz}{dw} = -i[t + \sqrt{(t^2 - 1)}]$$

are the two relations from which w in terms of z is to be derived.

Along the stream line $\psi = 0$,

$$\phi = E/t^2,$$

$$\frac{dx}{d\phi} + i\frac{dy}{d\phi} = \frac{dz}{d\phi} = -it - i\sqrt{(t^2 - 1)},$$

$$\therefore \frac{dx}{d\phi} = 0 \text{ and } \frac{dy}{d\phi} = -it - i\sqrt{(t^2 - 1)} \text{ along } OA,$$

$$\therefore OA = -i\int_1^\infty d\phi[t + \sqrt{(t^2 - 1)}] = -2Ei\int_1^\infty \frac{dt}{t^3}[t + \sqrt{(t^2 - 1)}],$$

and the coefficient E is at once expressible in terms of the breadth of the plane, giving in fact

$$E = l/(\pi + 4),$$

where $2l =$ total breadth. Along the free stream line,

$$1 > t > 0, \quad \log\frac{dz}{d\phi} = \log[t + i\sqrt{(1-t^2)}] - i\pi/2$$
$$= i[\tan^{-1}\sqrt{(1-t^2)}/t - \pi/2]$$
$$= \log 1/q + i\theta,$$
$$\therefore\ \theta = \tan^{-1}\sqrt{(1-t^2)}/t - \pi/2,$$

giving
$$t = \sin\theta.$$

Moreover
$$q = 1 = \frac{d\phi}{ds},$$

and
$$\phi = l/(\pi+4)\,t^2 = l/(\pi+4)\sin^2\theta,$$

and the intrinsic equation to the free surface is

$$s = \phi = l/(\pi+4)\sin^2\theta,$$

measuring s from the centre of the plate.

From these equations the expressions for x and y are easily derived, viz.

$$x = \frac{l}{\pi+4}(\cot\theta\operatorname{cosec}\theta - \log\cot\theta/2),$$
$$y = \frac{2l}{\pi+4}(1 - \operatorname{cosec}\theta),$$

from which the free stream line may be easily plotted. Both x and y become infinite together.

§ **17.** *Calculation of Thrust on Plane.* On any element dy of the plane the pressure on the up-stream side is $p_0 + \frac{\rho}{2}(V^2 - q^2)$, and p_0 on the down-stream side where $V =$ velocity of the steady stream at infinity, taken in this case to be unity.

Hence

$$\text{Total thrust} = 2\int_\infty^1 \left[p_0 + \frac{\rho}{2}(1-q^2) - p_0\right]dy = \rho\int_\infty^1 (1-q^2)\,dy$$
$$= \rho\int_\infty^1\left(\frac{1}{q} - q\right)\frac{d\phi}{dy}\cdot dy = \rho\int_\infty^1\left(\frac{1}{q} - q\right)\frac{d\phi}{dt}\,dt.$$

Now for this range of t we have found that

$$1/q = t + \sqrt{(t^2-1)},$$
$$\phi = l/(\pi+4)\,t^2.$$
$$\therefore\ \text{Thrust} = -\frac{4\rho l}{\pi+4}\int_\infty^1\sqrt{(t^2-1)}\,\frac{dt}{t^3} = \frac{\pi\rho l}{\pi+4}$$

and clearly if instead of assuming unit velocity at infinity, V be taken, the thrust on the lamina would be given by

$$T = \pi\rho l V^2/(\pi+4) \quad \text{.....................(20).}$$

§ 18. ***Symmetrical Flow past a Curved Boundary.*** **It has been shown in a paper by one of the authors* that by the addition of expressions for Ω for symmetrical flow past planes, cases of flow past curved barriers may be obtained. It will be sufficient to illustrate with the following example,**

$$\Omega = \log 1/q + i\theta = \tfrac{1}{2}\log[t + \sqrt{(t^2 - 1)}] + \tfrac{1}{2}\log\left[\frac{t + \sqrt{(t^2 - 4)}}{2}\right].$$

Consider the intervals for t,

$+\infty > t > 2$,

$$1/q = [\sqrt{(t+1)} + \sqrt{(t-1)}][\sqrt{(t+2)} + \sqrt{(t-2)}]/2\sqrt{2},$$
$$\theta = 0.$$

$2 \geqslant t \geqslant 1$,

$$\log 1/q + i\theta = \tfrac{1}{2}\log[t + \sqrt{(t^2 - 1)}] + \tfrac{1}{2}\log\left[\frac{t + i\sqrt{(4 - t^2)}}{2}\right],$$
$$\therefore\ 1/q = [\sqrt{(t+1)} + \sqrt{(t-1)}]/2,$$
$$\theta = \tfrac{1}{2}\tan^{-1}\sqrt{(4 - t^2)}/t.$$

$1 \geqslant t \geqslant 0$,

$$\log 1/q + i\theta = \tfrac{1}{2}\log[t + i\sqrt{(1 - t^2)}] + \tfrac{1}{2}\log\left[\frac{t + i\sqrt{(4 - t^2)}}{2}\right],$$
$$\therefore\ q = 1,$$
$$\theta = \tfrac{1}{2}\tan^{-1}\sqrt{(1 - t^2)}/t + \tfrac{1}{2}\tan^{-1}\sqrt{(4 - t^2)}/t, \text{ etc.}$$

It follows that for the range $+\infty > t \geqslant 2$ and $-2 \geqslant t > -\infty$ the boundary is flat, but for $2 \geqslant t \geqslant 1$ and $-1 \geqslant t \geqslant -2$ it is curved, θ being given in the range by

$$t = 2\cos 2\theta.$$

At $t = 1$, $\qquad \theta = 30°.$

The expression for w is, as in the preceding example,

$$w = E/t^2.$$

The form of the boundary and free stream line is shown in fig. 35, and the total thrust on the surface may be derived by exactly the same process as that previously adopted, care being taken to integrate the pressure components when dealing with the curved portion of the surface.

Since the fluid in the rear of this curved surface is at rest

* See footnote, page 50.

the nature of the flow will be totally independent of the shape of the surface lying within this region, and consequently the above analysis treats the case of the "discontinuous" motion past a

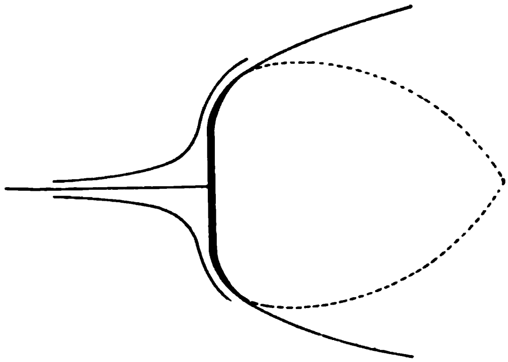


FIG. 35

strut. The fact that in this case the flow and accordingly also the resistance is unaffected by the shape and length of "tail" should however be specially noticed and will be referred to later.

§ **19.** *Flow past an Inclined Plane.* Let IO be the stream line $\psi = 0$ which impinges on the inclined plane at O where the velocity is zero, and divides along OBD and OAC, and suppose the values of t distributed as in fig. 36, where α is the inclination

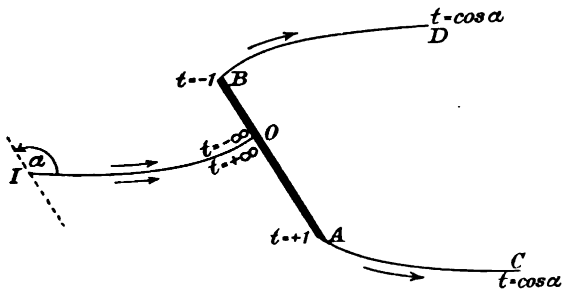


FIG. 36

of the plane to the direction of the fluid at I. From a consideration of the corresponding w- and w^{-1}-plane as in the previous case it is easily found that

$$w = E/(t - \cos\alpha)^2.$$

Along OA, $\theta = \alpha - \pi$ and q ranges from 0 to 1.
Along AC, q is unity and θ ranges from $\alpha - \pi$ to 0.
Along OB, $\theta = \alpha$ and q ranges from 0 to 1.
Along BD, q is unity and θ ranges from α to 0.

The corresponding Ω-plane is thus fig. 37.

Applying the Schwarz-Christoffel theorem the same formula for transformation as before is derived, viz.

$$\Omega = C \log [t + \sqrt{(t^2 - 1)}] + D,$$

where the constants C and D are now to be selected to satisfy the new conditions.

$+ \infty > t \geqslant 1$,

$$\log 1/q + i\theta = C \log [t + \sqrt{(t^2 - 1)}] + D.$$

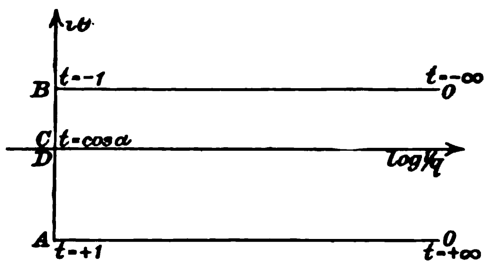


FIG. 37

At A, $\quad t = 1, \quad q = 1, \quad \theta = (\alpha - \pi)$,

$$\therefore\ D = (\alpha - \pi)\, i.$$

$1 > t > 0$,

$$\log 1/q + i\theta = C \log [t + i\sqrt{(1 - t^2)}] + D$$
$$= Ci \tan^{-1} \sqrt{(1 - t^2)}/t + (\alpha - \pi)\, i.$$

$0 > t > -1$,

$$\log 1/q + i\theta = C \log [-t - i\sqrt{(1 - t^2)}] + C\pi i + D$$
$$= C \tan^{-1} \sqrt{(1 - t^2)}/t + C\pi i + (\alpha - \pi)\, i.$$

$-1 \geqslant t > -8$,

$$\log 1/q + i\theta = C \log [-t + \sqrt{(t^2 - 1)}] + C\pi i + (\alpha - \pi)\, i.$$

At B, $\quad t = -1, \quad q = 1, \quad \theta = \alpha$,

$$\therefore\ C = 1.$$

Hence finally

$$\log \frac{dz}{dw} = \Omega = \log [t + \sqrt{(t^2 - 1)}] + (\alpha - \pi)\, i,$$

$$\therefore \frac{dz}{dw} = (\cos\alpha - i\sin\alpha)\,[t + \sqrt{(t^2 - 1)}],$$

and
$$w = E/(t - \cos\alpha)^2.$$

Along the stream-line $\psi = 0$,

$$\phi = E/(t - \cos\alpha)^2,$$

$$\frac{dx}{d\phi} + i\frac{dy}{d\phi} = (\cos\alpha - i\sin\alpha)\,[t + \sqrt{(t^2 - 1)}].$$

Hence along OA,

$$\frac{dx}{d\phi} = \cos\alpha\,[t + \sqrt{(t^2 - 1)}],$$

and
$$\frac{dy}{d\phi} = -\sin\alpha\,[t + \sqrt{(t^2 - 1)}].$$

By integration, the coefficient E is immediately expressible in terms of OA.

Similarly along OB,

$$\frac{dx}{d\phi} + i\frac{dy}{d\phi} = (\cos\alpha - i\sin\alpha)\,[t - \sqrt{(t^2 - 1)}],$$

$$\therefore \frac{dx}{d\phi} = \cos\alpha\,[t - \sqrt{(t^2 - 1)}],$$

and
$$\frac{dy}{d\phi} = \sin\alpha\,[t - \sqrt{(t^2 - 1)}],$$

and by integration, a relation is obtained between E and OB. Taking as before the breadth of the plane as $2l$ this provides

$$E = -\frac{l\sin^4\alpha}{\pi\sin\alpha + 4}$$

and determines the distance of O, the point at which the fluid divides, from the edge.

To derive the resultant pressure normal to the plane it is necessary to integrate

$$\tfrac{1}{2}\rho\,(1 - q^2)\,ds \quad \text{or} \quad \rho\,(1/q - q)\,q\,dx/2\cos\alpha$$

over the plane. The process is exactly analogous to that already given for the case of $\alpha = \pi/2$ and gives as result

$$T = \frac{l\pi\rho\sin\alpha}{\pi\sin\alpha + 4}\cdot V^2 \quad\ldots\ldots\ldots\ldots\ldots\ldots(21),$$

where V is the velocity at infinity.

The lift and drag is then derived directly by taking the components in the respective directions.

§ 20. ***Flow past a Cambered Plane.*** **It has been shown that problems dealing with the impact of a fluid against a symmetrical body, such as a strut, may be derived by the appropriate addition of two expressions for Ω applicable to cases of the impact on a plane. By a similar means solutions of problems of the flow past cambered planes may be constructed.**

Consider two expressions for Ω applicable to the flow past planes where B successively corresponds to $t = -1$ and to $t = -3$ say. These give

$$\Omega_1 = C_1 \log [t + \sqrt{(t^2 - 1)}] + D_1$$

and $$\Omega_2 = C_2 \log [t + 1 + \sqrt{(t - 1)(t + 3)}] + D_2$$

respectively.

As a particular case consider

$$\Omega = \tfrac{1}{2} \log [t + \sqrt{(t^2 - 1)}] + \tfrac{1}{2} \log [t + 1 + \sqrt{(t - 1)(t + 3)}] + i(\pi - \alpha)$$

$$= \log \left[\frac{\sqrt{(t+1)} + \sqrt{(t-1)}}{\sqrt{2}}\right] + \log \left[\frac{\sqrt{(t-1)} + \sqrt{(t+3)}}{2}\right] + i(\pi - \alpha)$$

with the same expression as before for w, viz. $w = E/(t - \cos \alpha)^2$. Then

$+\infty > t \geqslant 1$,

$$1/q = [\sqrt{(t + 1)} + \sqrt{(t - 1)}][\sqrt{(t - 1)} + \sqrt{(t + 3)}]/2\sqrt{2},$$
$$\theta = \pi - \alpha,$$

and therefore this portion of the plane is not cambered.

$1 \geqslant t \geqslant -1$,

$$\Omega = \log\left[\frac{\sqrt{(t+1)} + i\sqrt{(1-t)}}{\sqrt{2}}\right] + \log\left[\frac{i\sqrt{(1-t)} + \sqrt{(t+3)}}{2}\right] + i(\pi - \alpha),$$

$$\therefore \theta = \left[\tan^{-1}\sqrt{\frac{1-t}{1+t}} + \tan^{-1}\sqrt{\frac{1-t}{3+t}} + \pi - \alpha\right],$$

$$q = 1,$$

which corresponds to the free stream line.

$-1 \geqslant t \geqslant -3$,

$$\Omega = \log\left[\frac{i\sqrt{(-t-1)} + i\sqrt{(1-t)}}{\sqrt{2}}\right] + \log\left[\frac{i\sqrt{(1-t)} + \sqrt{(t+3)}}{2}\right] + i(\pi - \alpha)$$

$$= \log\left[\frac{\sqrt{(-t-1)}+\sqrt{(1-t)}}{\sqrt{2}}\right] + \log i + i\tan^{-1}\frac{\sqrt{(1-t)}}{\sqrt{(t+3)}} + i(\pi-\alpha),$$

$$\therefore\ 1/q = [\sqrt{(-t-1)}+\sqrt{(1-t)}]/\sqrt{2},$$

$$\theta = 3\pi/2 - \alpha + \tan^{-1}\sqrt{(1-t)}/\sqrt{(t+3)},$$

giving the velocity and inclination at each point of the cambered portion of the plane.

$-3 \geqslant t > -\infty$,

$$\Omega = \log\left[\frac{i\sqrt{(-t-1)}+i\sqrt{(1-t)}}{\sqrt{2}}\right] + \log\left[\frac{i\sqrt{(1-t)}+i\sqrt{(-t-3)}}{2}\right] + i(\pi-\alpha),$$

$$\therefore\ 1/q = [\sqrt{(-t-1)}+\sqrt{(1-t)}][\sqrt{(1-t)}+\sqrt{(-t-3)}]/2\sqrt{2},$$

$$\theta = 2\pi - \alpha,$$

showing that the remainder of the plane is uncambered.

Fig. 38 shows the cambered plane, and the nature of the flow past it. By an alteration of the constants occurring in the

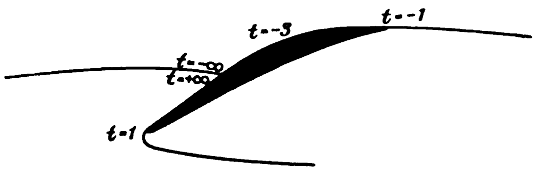


FIG. 38

expression for Ω, problems relating to planes cambered in the opposite sense and situated at any angle of attack to the direction of the fluid motion at infinity may be solved.

§ **21.** When the surface is concave to the fluid so that the "dead water" region is in the rear and upwards from the plane, a positive lift will of course be produced, but it is clear that the shape of the upper surface of the aerofoil will in that case have no effect whatsoever on the nature of the motion and consequently on the forces exerted. Now experiments on aerofoils in air have undoubtedly shown that the upper surface and particularly certain portions of it exercise a most potent effect on the nature of the flow and the forces brought into existence, while the effect of the under surface is extremely insensitive to changes in shape. A similar conclusion has already been drawn regarding the complete unimportance of the rear portion of a strut in the case of the

"discontinuous" flow of a perfect fluid, whereas in actual reality that portion of the strut plays an extremely important part, on account of the steadying influence it exerts upon the flow. It would appear, therefore, that although surfaces acted upon by a perfect fluid flowing in the manner here supposed do experience a resistance, that resultant thrust is totally uninfluenced by the very factors that regulate it in the case of a real fluid. In a number of other respects a vital difference exists. Far from the region in the rear of the body being one of calm "dead water," it is in fact generally one of violent turbulence and as such is equivalent to a dissipation of energy that finds no place in the problem discussed. A free surface, moreover, involving as it does a discontinuity in velocity over its boundary, is in reality a vortex sheet, and consequently is highly unstable, tending to roll itself up at various points and thus form a series of isolated vortices or eddies. The exact nature and action of these will be discussed later. In spite of all these objections nevertheless it is remarkable, even admitting the assumptions, that the results obtained for the resistance are comparable at all with those derived by experiment, although deficient in the respects indicated. A considerable volume of discussion among mathematical physicists has centred round the question of the applicability of the state of discontinuous motion even to a perfect fluid. It is clear that the state of affairs supposed existing would not originate in a fluid in a finite time since it would necessarily, if allowed, curl closely round both sides of the obstacle. The motion may, however, be gradually approximated to, by supposing a surface of exactly the dimensions and curvature of the calculated free stream lines to occupy its position and the fluid allowed to flow past it and the body. When the motion has attained steadiness, if the surface be supposed removed without the introduction of any disturbance, a state of discontinuous flow will have been brought into existence. It may be mentioned that the method of this section finds a legitimate application however in the theory of jets.

§ **22.** The only other serious attempt to originate a reaction between a body and a perfect fluid is that strongly advocated by Mr F. W. Lanchester, among others, in which he supposes a cyclic motion, about the aerofoil say, superposed upon the ordinary steady streaming. As a consequence the flow is retarded

on the one side of the body and accelerated on the other. Since this involves an increased and a diminished pressure respectively on opposite sides, a resultant force will be originated. It can be shown that this force is proportional to the intensity of the cyclic motion and is thus apparently quite arbitrary. The method moreover gives rise to a lift on the aerofoil and no drag. It is not proposed to enter here into a discussion of this method for which the student is referred to *Aerofoil and Propeller Theory* by F. W. Lanchester.

The failure of the various treatments of the problem of the motion of a body and the forces experienced to approximate to that of practice is evidently due to the supposition that the fluid dealt with is perfect. The mathematical treatment of a viscous fluid is exceedingly intricate and merely the fringe will be touched in the present chapter. Nevertheless, the value of investigations of the behaviour of a perfect fluid is considerable for the light it throws upon the nature of the so-called "stream line" motion, and the ideal, never attainable, towards which to strive in design.

Rotational or Vortex Motion in a Perfect Fluid

§ 23. Attention has so far been confined to the problem of the steady motion of a perfect fluid devoid of vorticity, and there remains for discussion the second class of cases in which this restriction is no longer imposed. In §§ 6 and 7 it was seen that the intensity of spin, or the instantaneous angular rotation at any point, was measured by the expression

$$\nabla^2\psi = f(x, y),$$

or, as it shall be referred to here,

$$\nabla^2\psi = 2\zeta \quad \text{..........................(22).}$$

The function ζ is not restricted in any way. It may be a continuous function of position or it may exist only at discrete points, but it must be noticed that any region of the fluid in which ζ does not exist may be treated as irrotational, for which there will exist a function ϕ. It is not possible within the limits of the present chapter to prove rigorously many of the laws that govern vorticity in a perfect fluid, and it will not be attempted, but it will suffice for the present simply to state those laws that may throw light on the present question. For a rigorous proof of these the student

is referred to Lamb's *Hydrodynamics*. The following apply generally to three dimensions.

(*a*) A vortex line—a line in the fluid coinciding at each point with the axis of rotation there—cannot begin or end at any point of the interior of a fluid. Any vortex lines which exist must either form closed curves or else traverse the fluid beginning and ending on its boundaries. This presupposes that the velocities are continuous.

(*b*) Vortex lines move with the fluid.

(*c*) The strength of a vortex—twice the product of the angular velocity and the cross section—is constant with respect to time, if a potential function exists for the external forces operating on the fluid.

(*d*) The motion of an incompressible fluid which fills all space, at rest with infinity, is determinate when the value of the vorticity is known at all points.

§ **24.** To commence with the simplest problem in two dimensions consider the case where there is no vorticity in any part of the field except at the origin of coordinates at which point there exists a single vortex of constant strength K.

Since all directions are identical in relation to the vortex, the motion at any point and the function determining it, ψ, must depend solely on the distance r from the origin.

Writing $$r^2 = x^2 + y^2,$$
we find that

$$2\zeta = \nabla^2\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = \frac{d^2\psi}{dr^2} + \frac{1}{r}\frac{d\psi}{dr} = \frac{1}{r}\left(r\frac{d^2\psi}{dr^2} + \frac{d\psi}{dr}\right)$$

$$= \frac{1}{r}\frac{d}{dr}\left(r\frac{d\psi}{dr}\right).$$

$$\therefore \frac{d}{dr}\left(r\frac{d\psi}{dr}\right) = 2\zeta r \quad \text{.............................(23).}$$

If a be the radius of the vortex $2\pi a^2\zeta$ will be its total strength,

$$\therefore\ 2\pi a^2\zeta = K \text{ from } r = 0 \text{ to } r = a,$$

$$\zeta = 0 \text{ from } r = a \text{ to } r = \infty.$$

Integrating equation (23) it follows that

$$r\frac{d\psi}{dr} = \zeta r^2 + A,$$

$$\therefore \frac{d\psi}{dr} = \zeta r + \frac{A}{r} \quad \text{........................(24).}$$

(i) Outside the vortex $\zeta = 0$,

$$\therefore \frac{d\psi}{dr} = \frac{A}{r}.$$

$$\therefore \psi = A \log r + B.$$

If $\psi = 0$ when $r = a$,

$$\psi = A \log r/a.$$

The velocity at any point is then

$$u = -\frac{\partial\psi}{\partial y} = -\frac{A}{r}\frac{\partial r}{\partial y} = -\frac{A}{r}\cdot\frac{y}{r},$$

$$v = \frac{\partial\psi}{\partial x} = \frac{A}{r}\frac{\partial r}{\partial x} = \frac{A}{r}\cdot\frac{x}{r}.$$

At $r = a$, $u = -Ay/a$, $v = Ax/a^2$.

(ii) Inside the vortex ζ is constant; integrating (24)

$$\psi = \zeta r^2/2 + A_1 \log r + B_1$$
$$= \zeta (r^2 - a^2)/2 + A_1 \log r/a,$$

since $\psi = 0$ when $r = a$.

Since ψ is not to be infinite when $r = 0$, A_1 must be zero.

$$\therefore \psi = \zeta (r^2 - a^2)/2,$$
$$\therefore u = -\zeta r y/r = -\zeta y,$$
$$v = +\zeta r x/r = \zeta x.$$

In order that the velocity at the boundary of the vortex may be continuous it follows that

$$\zeta y = Ay/a^2, \quad \zeta x = Ax/a^2,$$
$$\therefore A = \zeta a^2 = K/2\pi.$$

Accordingly the stream function outside the vortex becomes

$$\psi = \zeta a^2 \log r/a = \frac{K}{2\pi} \log r/a,$$

and the component velocities

$$u = -\frac{\zeta a^2}{r}\cdot\frac{y}{r}, \quad v = \frac{\zeta a^2}{r}\cdot\frac{x}{r}, \quad q = \frac{d\psi}{dr} = \frac{\zeta a^2}{r} = \frac{K}{2\pi}\cdot\frac{1}{r}.$$

The velocity is thus at any point directed perpendicular to the radius vector and of magnitude $K/2\pi$ times the inverse distance from the vortex (see fig. 39), and the stream lines are of course circles round the vortex as centre. The stream function due to a single vortex at the origin is

$$\psi = K/2\pi \,.\, \log r,$$

omitting the constant term, may be represented in a form more convenient for further development. Since apart from the origin no vorticity exists in the fluid, for all other points in the field a potential function must also exist. This is of course the function conjugate to the above and is easily derived from it. It has already been seen that

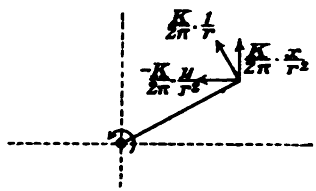


FIG. 39

$$\log z = \log (x + iy) = \log re^{i\theta} = \log r + i\theta.$$

The function conjugate to $\log r$ is thus θ, and accordingly

$$w = \phi + i\psi = iK/2\pi \,.\, \log z$$

gives the relation between w and z for the fluid outside the origin.

Generally

$$w = iK/2\pi \,.\, \log (z - z_0) \quad \ldots\ldots\ldots\ldots\ldots\ldots(25)$$

is the solution for a vortex situated at the point

$$z_0 = x_0 + iy_0.$$

§ **25.** From this single illustration may be derived the nature of the motion in any case in which a number of vortices of this nature are situated at given positions in the fluid. Two principles are utilised in such a determination. It has been shown earlier in this chapter that when the motion of the fluid is determined by linear differential equations of the type here involved that the stream functions for any number of possible motions may be added and a new type of possible flow derived such that the velocity at any point is simply the vectorial sum of the component velocities due to each of the superposed types of flow. This conception will be here applied to the case where any number of discrete vortices exist, by superposing on any point the component motions due to each of the vortices separately. In the second place, since these vortices must move with the fluid, any single vortex must have imparted to it the motions at that point due to all the others. The method will become clear by considering the case of two vortices A and A' of equal strengths but opposite signs situated at the points $x = 0$, $y = a$ and $x = 0$, $y = -a$ respectively. A' imparts a velocity $\frac{K}{2\pi} \cdot \frac{1}{2a}$ to the fluid occupying the position A, while the latter likewise causes A' to move forward

with the same velocity. The whole system thus moves forward with constant speed $K/4\pi a$ in the positive direction. The stream function and consequently the stream lines also are easily deduced from the analysis of the last example.

For the vortex at A,

$$w = iK/2\pi \,.\, \log(z - ai),$$

and for A',

$$w = -\, iK/2\pi \,.\, \log(z + ai).$$

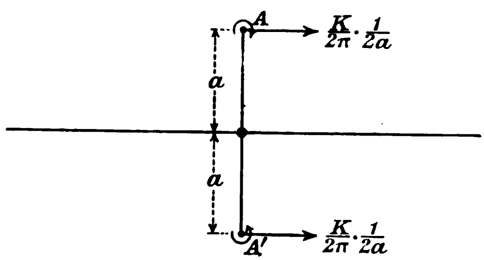


FIG. 40

Superposing these two systems the stream function for the final state of motion is given by

$$\phi + i\psi = w = \frac{iK}{2\pi}\log(z - ai) - \frac{iK}{2\pi}\log(z + ai) = \frac{iK}{2\pi}\log\frac{z - ai}{z + ai}.$$

Hence

$$\psi = \frac{K}{4\pi}\log\frac{x^2 + (y - a)^2}{x^2 + (y + a)^2}.$$

In order to obtain the stream lines corresponding to a system of axes continually occupying the same position relative to the two vortices, that is, moving with them, it will be necessary to

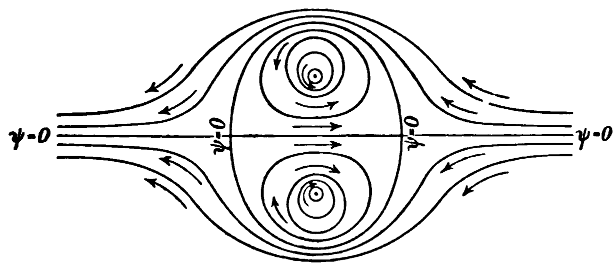


FIG. 41

add to this stream function such an expression as will involve the superposition of a horizontal velocity $-\frac{K}{2\pi}\cdot\frac{1}{2a}$ upon the whole field of motion.

Such an expression is $Ky/4\pi a$.

Hence finally

$$\psi = \frac{K}{4\pi} \log \frac{x^2 + (y-a)^2}{x^2 + (y+a)^2} + \frac{Ky}{4\pi a} \quad \text{............(26).}$$

An analysis of the system of curves given by this equation for a series of values of ψ indicates that they are of the nature shown in fig. 41. The stream line $\psi = 0$ consists of the X-axis and an oval enclosing the two vortices, moving with them and continually enclosing the same fluid.

§ **26.** Following the method thus indicated the more complicated case of two pairs of vortices and their mutual effect can be easily developed. The detailed calculation of the stream lines in this case (fig. 42) is left to the student but the general motion of the vortices can be easily seen.

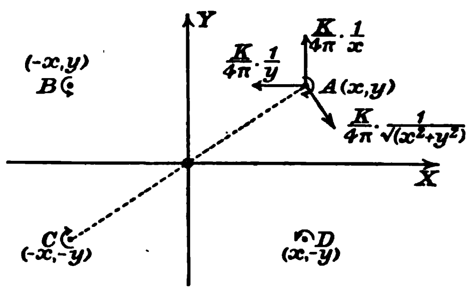


Fig. 42

If the origin be situated symmetrically between the four vortices K at (x, y), $-K$ at $(-x, y)$, K at $(-x, -y)$, and $-K$ at $(x, -y)$, say A, B, C, D respectively, then the velocities imparted to A by the remaining three are as indicated above.

If $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are the component velocities of A, then

$$\frac{dx}{dt} = -\frac{K}{4\pi}\cdot\frac{1}{y} + \frac{K}{4\pi}\cdot\frac{1}{\sqrt{(x^2+y^2)}}\cdot\frac{y}{\sqrt{(x^2+y^2)}} = -\frac{K}{4\pi}\cdot\frac{x^2}{y(x^2+y^2)},$$

and

$$\frac{dy}{dt} = \frac{K}{4\pi}\cdot\frac{1}{x} - \frac{K}{4\pi}\cdot\frac{1}{\sqrt{(x^2+y^2)}}\cdot\frac{x}{\sqrt{(x^2+y^2)}} = \frac{K}{4\pi}\cdot\frac{y^2}{x(x^2+y^2)}.$$

Hence $$\frac{dy}{dx} = -\frac{y^3}{x^3}.$$

On integration this furnishes the path traced out by the vortex as

$$1/x^2 + 1/y^2 = 1/a^2.$$

It appears at once from this that the two pairs of vortices approach each other asymptotically, each moving meanwhile further away from its partner.

It must be noticed that OX and OY from conditions of symmetry are evidently stream lines and may therefore be taken as rigid boundaries. The above analysis thus gives the path traced out by a vortex in the vicinity of a corner, or alternatively the manner of approach of a pair of vortices to a plane surface.

The student will find it an interesting exercise to trace the mutual effect of such a double pair when those at B and C are identical in strength and sign with those at A and D respectively, the latter two being equal in magnitude but opposite in sign. The foremost pair it will be found widen while the pair behind come closer together and then shoot through the pair in front.

§ **27.** Photographs illustrative of the nature of the flow behind a moving body have been given in Chapter II. It is clear that for at least a short time after their production the eddies or vortices brought into existence in the immediate wake of the body apparently adopt and retain a very definite arrangement. By assuming a system of vortices situated similarly to those shown in the photographs mentioned, it will be seen, even in the case of fluid devoid of viscosity to which the present discussion is confined, that the nature of the flow and stream lines bears a close resemblance to that actually photographed.

It will be supposed that the vortices are situated in two infinite rows, all of strength $+K$ in the one row, and $-K$ in the other.

Let their positions be as indicated below.

The values of $z = x + iy$ for the eddies are then

(1) Top row, of strength $+K$,

$$(A_0, hi) : (A, \ 2l + hi) : (B, \ 4l + hi) : \ldots,$$
$$(A', -2l + hi) : (B', -4l + hi) : \ldots,$$

and

(2) Bottom row, of strength $-K$,

$$(\alpha, \ l - hi) : (\beta, \ 3l - hi) : \ldots,$$
$$(\alpha', -l - hi) : (\beta', -3l - hi) : \ldots.$$

Remembering that the relation between z and w for a single eddy at $z = z_0$ is

$$w = \frac{iK}{2\pi} \log (z - z_0) \quad \text{.....................(25)},$$

it follows that the complete expression for this case is

$$\begin{aligned}
w = \frac{iK}{2\pi} [&\log (z - hi) + \log (z - 2l - hi) + \log (z - 4l - hi) \dots \\
&+ \log (z + 2l - hi) + \log (z + 4l - hi) \dots \\
&- \log (z - l + hi) - \log (z - 3l + hi) \dots \\
&- \log (z + l + hi) - \log (z + 3l + hi) \dots] \\
= \frac{iK}{2\pi} \{&\log (z - hi) + \log [(z - hi)^2 - 4l^2] + \log [(z - hi)^2 - 16l^2] \dots \\
&- \log [(z + hi)^2 - l^2] - \log [(z + hi)^2 - 9l^2] \dots\} \\
= \frac{iK}{2\pi} &\log \frac{(z - hi)[(z - hi)^2 - 4l^2][(z - hi)^2 - 16l^2] \dots}{[(z + hi)^2 - l^2][(z + hi)^2 - 9l^2] \dots}.
\end{aligned}$$

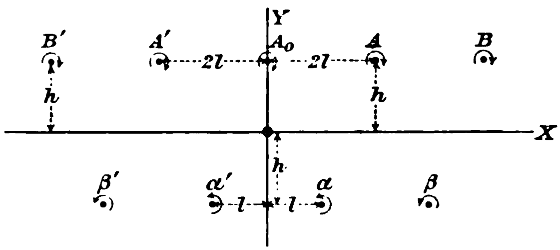


FIG. 43

For simplicity let $\frac{z - hi}{l} = p$ and $\frac{z + hi}{l} = q$, then the above formula may be written

$$w = \frac{iK}{2\pi} \log \frac{p\left(1 - \frac{p^2}{2^2}\right)\left(1 - \frac{p^2}{4^2}\right) \dots}{\left(1 - \frac{q^2}{1^2}\right)\left(1 - \frac{q^2}{3^2}\right) \dots} + \text{constant}.$$

Now it is well known that $\sin \Theta$ and $\cos \Theta$ may be expressed as products in the following form:

$$\sin \Theta = \Theta \left(1 - \frac{\Theta^2}{\pi^2}\right)\left(1 - \frac{\Theta^2}{2^2\pi^2}\right) \dots,$$

$$\cos \Theta = \left(1 - \frac{2^2\Theta^2}{1^2\pi^2}\right)\left(1 - \frac{2^2\Theta^2}{3^2\pi^2}\right) \dots.$$

Hence $p\left(1-\frac{p^2}{2^2}\right)\left(1-\frac{p^2}{4^2}\right)\ldots=\frac{\sin p\pi/2}{\pi/2}=\frac{2}{\pi}\sin\frac{\pi(z-hi)}{2l}$,

and $\left(1-\frac{q^2}{1^2}\right)\left(1-\frac{q^2}{3^2}\right)\ldots=\cos q\pi/2=\cos\frac{\pi(z-hi)}{2l}$.

The relation between w and z which furnishes all the circumstances of the motion is thus, omitting the constant terms,

$$\phi+i\psi=w=\frac{iK}{2\pi}\log\frac{\sin\frac{\pi}{2l}(z-hi)}{\cos\frac{\pi}{2l}(z+hi)}.$$

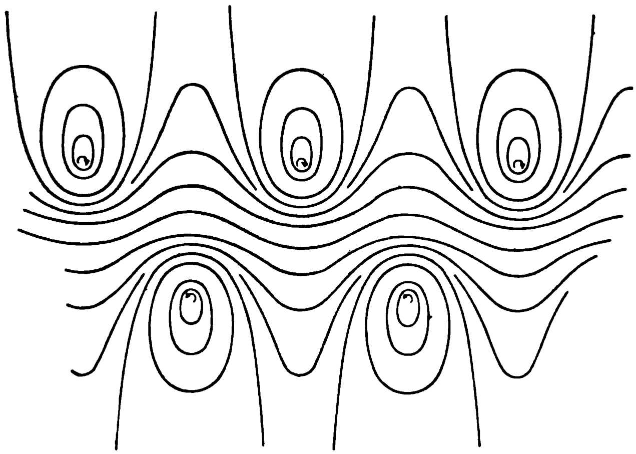


FIG. 44

Separating the real from the imaginary parts it follows that the stream lines are given by

$$\psi=\frac{K}{2\pi}\log\frac{\cosh\frac{\pi}{l}(y-h)-\cos\frac{\pi x}{l}}{\cosh\frac{\pi}{l}(y+h)+\cos\frac{\pi x}{l}}\quad\ldots\ldots\ldots\ldots(27).$$

The form of the stream lines is shown in fig. 44.

VISCOSITY

§ **28.** All real fluids exhibit in varying degree a resistance to distortion known as viscosity. The significance of this internal force from the molecular point of view has already been brought

out in the previous chapter, and it is here proposed to indicate how the nature of the flow of such fluids may be investigated exactly in a few simple cases. The solution of the general equations of the motion of a viscous fluid, subject to given boundary conditions, would clarify at one stroke the whole problem of aerodynamics, but, unfortunately, mathematical theory has not yet been sufficiently developed to effect this. Two facts already discussed at some length will be recalled.

(1) If $\partial v/\partial n$ be the velocity gradient perpendicular to the stream lines at any point, and if F be the viscous dragging force existing there per unit area, then

$$F = \mu \frac{\partial v}{\partial n},$$

where $\mu =$ coefficient of viscosity.

It is more customary and convenient to deal with the coefficient of kinematic viscosity $= \nu = \mu/\rho$.

(2) Experience appears to indicate that at the surface of contact between a solid and a viscous fluid there is no relative motion of the solid and fluid in immediate contact with it, otherwise an infinitely greater resistance would require to be opposed to the sliding of one portion of the fluid past another than to the passage of the fluid over the solid.

§ **29.** *Steady motion of a viscous fluid under pressure between two fixed parallel walls.* Suppose the fluid flowing steadily between the walls AB and CD, in the direction OX, and consider the forces upon any small element $\delta x\, \delta z$. The resultant pressure thrust in the direction OX is

$$p\,\delta z - \left(p + \frac{\partial p}{\partial x}\,\delta x\right)\delta z = -\frac{\partial p}{\partial x}\,\delta x\,\delta z.$$

If u be the velocity at the element, the positive traction due to viscous forces on the under surface of the element is $\mu \dfrac{\partial u}{\partial z}\,\delta x$ and that on the upper surface

$$-\mu\left(\frac{\partial u}{\partial z} + \frac{\partial^2 u}{\partial z^2}\,\delta z\right)\delta x.$$

Hence the total viscous dragging force in the direction OX is

$$+\mu\frac{\partial u}{\partial z}\,\delta x - \mu\left(\frac{\partial u}{\partial z} + \frac{\partial^2 u}{\partial z^2}\,\delta z\right)\delta x = -\mu\frac{\partial^2 u}{\partial z^2}\,\delta z\,\delta x.$$

Since the motion is steady the pressure and viscous forces alone determine the motion within the given boundaries

$$-\frac{\partial p}{\partial x}\,\delta x\,\delta z = -\mu\frac{\partial^2 u}{\partial z^2}\,\delta z\,\delta x,$$

$$\frac{\partial p}{\partial x} = \mu\frac{\partial^2 u}{\partial z^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(28).$$

The corresponding equations for motion in the directions OZ and OY give

$$\frac{\partial p}{\partial y} = 0, \quad \frac{\partial p}{\partial z} = 0,$$

and accordingly since the motion is steady so that the velocity is not a function of x, the pressure gradient $\partial p/\partial x$ must be an absolute constant,

$$\therefore \mu\frac{\partial^2 u}{\partial z^2} = 2A,$$

$$\therefore u = \frac{A}{\mu}(z^2 - h^2),$$

making the velocity zero at $z = \pm h$, the sides of the channel.

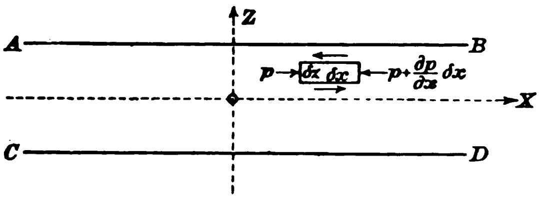


FIG. 45

If M be the total quantity of fluid crossing any section per second,

$$M = \rho\int_{-h}^{h} u\,dz = -\frac{4Ah^3}{3\mu}.$$

Hence finally $$u = \frac{3M}{4\rho h^3}(h^2 - z^2)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(29).$$

This formula determining the distribution of velocity across the channel has received ample verification mainly where the fluid is water, and where the conditions regarding steadiness are those assumed in the foregoing discussion. Actually for a channel of given dimensions this state of steadiness in flow cannot be attained unless the speed lies below a certain value known as the critical speed. This question has already been treated to some extent in the previous chapter and will be dealt with again in that on dynamical similarity.

§ **30.** *Steady flow along a circular pipe.* A similar problem amenable to equally simple treatment is that of the steady flow of a viscous fluid along a uniform circular pipe.

Let the axis of X be taken to coincide with that of the pipe and let r be the radius to a point in the fluid in a plane perpendicular to the axis. Consider the forces, viscous and pressure, acting upon a tube of the fluid of length δx, radius r and thickness δr. If u be the axial velocity at radius r then the viscous force per unit length on one side of the shell is $\mu \frac{\partial u}{\partial r}$ or round the whole shell $\mu \frac{\partial u}{\partial r} . 2\pi r$, while on the other side it is

$$\mu \frac{\partial u}{\partial r} . 2\pi r + \mu \frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r} . 2\pi r\right) \delta r.$$

Hence the resulting dragging force over the length δx is

$$2\pi\mu \frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r}\right) \delta x \, \delta r.$$

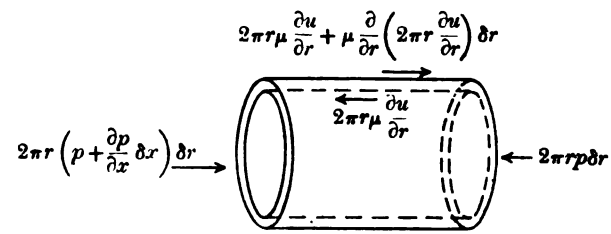


FIG. 46

The motion being steady, this must be balanced by the pressures on the two ends of the tubular shell, the resultant of which is clearly

$$2\pi r \delta r \frac{\partial p}{\partial x} \delta x.$$

Accordingly
$$\mu \frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r}\right) = - r \frac{\partial p}{\partial x} \quad \text{.......................(30).}$$

Since the motion is everywhere longitudinal there are no viscous forces radially and consequently the pressure variation in that direction must vanish and therefore $\partial p/\partial r$ must vanish. The left-hand side of equation (30) being a function of r only, $\partial p/\partial x$ must therefore be a constant. If p_1 and p_2 be the pressures at the ends of the pipe of length l,

$$\frac{\partial p}{\partial x} = \frac{p_1 - p_2}{l}.$$

Hence $$\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = -\left(\frac{p_1 - p_2}{\mu l}\right) r.$$

The integral of this equation is evidently

$$u = -\left(\frac{p_1 - p_2}{4\mu l}\right) r^2 + a \log r + b.$$

The constants of this equation are to be determined from the fact that there is no slip at the boundary and the velocity along the axis must be finite, i.e. $u = 0$ at $r = R$ and u finite at $r = 0$.

Hence $$u = \left(\frac{p_1 - p_2}{4\mu l}\right)(R^2 - r^2) \quad \text{..................(31)},$$

and the velocity distribution is apparently parabolic, the speed at the centre being

$$\left(\frac{p_1 - p_2}{4\mu l}\right) R^2.$$

§ **31.** *Steady flow under pressure between two concentric tubes.* The formula derived in the last example may be directly applied in this case.

$$u = -\left(\frac{p_1 - p_2}{4\mu l}\right) r^2 + a \log r + b.$$

The boundary conditions are now $u = 0$ at $r = R$ and $u = 0$ at $r = R_0$. Inserting these in the equation it is easily found that

$$a = \left(\frac{p_1 - p_2}{4\mu l}\right) \frac{R^2 - R_0^2}{\log R/R_0},$$

$$b = \left(\frac{p_1 - p_2}{4\mu l}\right) \left\{\frac{R_0^2 \log R - R^2 \log R_0}{\log R/R_0}\right\},$$

giving

$$u = \left(\frac{p_1 - p_2}{4\mu l}\right) \left[(R^2 - r^2) + (R^2 - R_0^2)\left\{\frac{\log r/R}{\log R/R_0}\right\}\right] \quad \text{.........(32)}.$$

An interesting particular case arises when the radius of the internal tube is vanishingly small, in which case, as the student will easily verify by a consideration of the limiting values of the expressions, the velocity distribution across the pipe is exactly that of the previous problem except in the immediate neighbourhood of the central wire where it falls steeply to zero.

The above formulae have been accurately verified by Poiseuille by experiments on the flow of water through capillary tubes. This is important from the present point of view from the fact

that it verifies in the case of that fluid the assumption here made that there is no slip at the boundary. The results can likewise be utilised for determining the coefficients of viscosity, for the total volume of fluid crossing any section per second in the case of a pipe of radius R is

$$\int_0^R 2\pi r u \, dr = \frac{\pi R^4 (p_1 - p_2)}{8\mu l}.$$

From a measurement of the quantity of fluid passing through the tube per second μ is at once calculable.

General Discussion and Derivation of Equations

§ **32.** For further discussion it will be convenient to revert to the method of treatment adopted in the earlier part of the previous chapter where the fluid instead of being regarded as a homogeneous medium is considered as composed of discrete particles or molecules shooting about in all directions. If the two methods of treatment are ultimately identical, the equations specifying the motion of the medium should be the same for both. There are certain aspects however of the motion of a viscous fluid such as air which can be brought out more strikingly by regarding them from the point of view of the molecular theory. It has already been explained that wherever there exists a velocity gradient in the medium there exists simultaneously a so-called viscous force in direction parallel to the flow and in magnitude proportional to the velocity gradient. The coefficient of proportionality under certain standard conditions was defined as the coefficient of viscosity.

If a body be brought under the existence of an external force of some nature forcing it through the medium, the particles coming into contact with it either directly or through the agency of intermediate particles will have communicated to them every second a definite quantity of momentum in the direction of the impressed force. This continuous exertion of an external force clearly performing work upon the fluid must likewise ultimately be equivalent to an increase in energy of the totality of the particles composing the fluid. Considering a small rectangular element enclosing a definite number of molecules, it is clear that the energy of this element considered as the sum of the energies of the molecules can in a sense be increased to an unlimited extent

without affecting its absolute momentum by simply superposing equal and opposite velocities upon two molecules of equal mass within the element. If no further external force be impressed upon the fluid beyond that exerted by the body already referred to, it is evident that the total quantity of momentum being emitted per second from that source will spread itself in all directions from molecule to molecule throughout the region of the fluid, its total magnitude remaining unchanged. It follows that if the body be in motion in a fluid of infinite extent so that an infinite quantity of fluid receives a finite increment in momentum per second, the fluid at a great distance will be undisturbed. Stating this in more exact form we may imagine a mass M per second of fluid at infinity receiving an increment in velocity δv per second so that $M \times \delta v$ is the momentum passed out per second from the body through the agency of the molecules and is finite. M being of infinite magnitude, δv is infinitely small. Consider the apparent energy this mass of fluid receives per second. It will be of the order $\frac{1}{2}M\,(\delta v)^2$ so that the total quantity of actual translational energy passed out per second at infinity is infinitely small. On the other hand it has already been explained that the body in its motion through the fluid is performing work at a finite rate which, on the basis of the law of conservation of energy, must ultimately make itself evident in some form, and the question naturally arises—where does this energy exist? Now it has been seen that each element of the fluid regarded as a whole may possess and transmit a certain quantity of momentum to its neighbour but may at the same time possess and accumulate energy regarded as the sum total of the energies of the individual molecules. In fact each element of the fluid acts as an agent to transmit momentum in a given direction to all the neighbouring elements and in the process without destroying any of that momentum increases its own energy. If the motion of the element is to be considered, as is done on the supposition of a homogeneous fluid already described, then the visible energy of translation will be merely a portion of this total energy and will be represented by $\frac{1}{2}mv^2$, where m is the mass of the element and v the average velocity in the general direction of motion. This omits the remainder of the internal energy existing in the molecules flying in all directions. It in fact takes no account of the so-called heat energy. It should now be clear that a body in its motion through a viscous fluid regarded

either from the molecular or homogeneous standpoint transmits energy to the fluid both in the translational and in the heat form. Ultimately the translational energy of any element becomes infinitely small and merely heat energy remains.

§ **33.** The vital distinction between a viscous fluid and a perfect fluid becomes at once apparent from the above remarks. The mere assumption, it has been seen, that the fluid is composed ultimately of discrete particles is sufficient to bring into existence a viscous dragging force. A perfect fluid where this force is non-existent must therefore be homogeneous throughout. The energy which would otherwise be secreted in the form of internal molecular energy, and consequently heat, no longer exists as such and simply passes on purely into the translational form as understood above. In the perfect fluid there is thus no apparent dissipation of energy.

§ **34.** *General equations of viscous flow.* The problem will be restricted as before to two dimensions. Consider a small element of fluid *abcd* of dimensions δx and δy moving with component

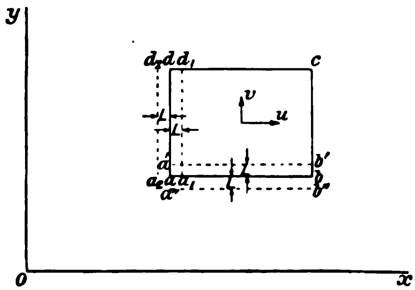


FIG. 47

velocities u along ox and v along oy, fig. 47. According to the kinetic theory of gases the molecules of the fluid are shooting about in all directions so that inside *abcd* the most probable velocity of any molecule in direction relative to the C.G. of the element is the same for all directions. It follows that if a mass m of molecules leaves the element through *unit* length of the side *ab* per second, the same mass will pass out through *unit* length of any other side, and be replaced by molecules of the same total mass from the fluid outside, assuming the density constant. Suppose that the molecules leaving through *ab* come from an average position *a′b′*,

distant L from ab, then the momentum per second lost by the element along ox would be $m\delta x\left(u' + L\dfrac{\partial u}{\partial y}\right)$, where u' is the average velocity along ab and $\partial u/\partial y$ the gradient in direction oy of the x component of the velocity, in the neighbourhood of the element. In the same way if the molecules entering the element across this face be supposed to come from $a''\ b''$, distant L from ab, where the x component of the velocity is $u' - L\dfrac{\partial u}{\partial y}$, the momentum gained per second by the element will be

$$m\delta x\left(u' - L\frac{\partial u}{\partial y}\right),$$

making a total loss per second through the transference across this face of $2mL\delta x\,(\partial u/\partial y)$. Now the average velocity of the fluid along cd is $u' + \dfrac{\partial u}{\partial y}\delta y$, so that the momentum gained per second by the element through the interchange of molecules over this face is

$$2mL\delta x\left\{\frac{\partial}{\partial y}\left(u' + \frac{\partial u}{\partial y}\delta y\right)\right\},$$

giving a resultant gain across the two faces of

$$2mL\delta x\delta y\,\frac{\partial^2 u}{\partial y^2}\text{ along } ox \quad \dots\dots\dots\dots\dots\dots(33),$$

since $\partial u'/\partial y = \partial u/\partial y$ nearly.

Proceeding on the same lines as before it is easily verified that the element will lose momentum at the rate of $2mL\dfrac{\partial u}{\partial x}\delta y$ per second in the direction ox by transference across ad which together with the gain across bc gives an increase of

$$2mL\delta x\delta y\,\frac{\partial^2 u}{\partial x^2}\text{ along } ox \quad \dots\dots\dots\dots\dots(34).$$

Adding this to (33) the net gain per second for the whole element is

$$2mL\delta x\delta y\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)\dots\dots\dots\dots\dots\dots(35).$$

The product $2mL$ is called μ, the coefficient of viscosity. The expression (35) giving the gain in momentum per second in the direction ox brought about by the molecular interchange is therefore equivalent to an external force per unit volume of magnitude $\mu\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}\right)$ and therefore the equations of

motion of the fluid will be the same as those of a perfect fluid except that

$$\rho X + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

must be substituted for ρX and

$$\rho Y + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \text{ for } \rho Y.$$

Hence if $\nu = \mu/\rho$, the kinematic viscosity, then

$$X - \frac{1}{\rho}\frac{\partial p}{\partial x} = \frac{\partial u}{\partial t} - \nu\nabla^2 u + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} \quad \text{............(36)},$$

$$Y - \frac{1}{\rho}\frac{\partial p}{\partial y} = \frac{\partial v}{\partial t} - \nu\nabla^2 v + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} \quad \text{............(37)},$$

where $$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

As before the equation of continuity requires

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \text{........................(38)}.$$

§ **35.** If no external forces are operating $X = Y = 0$, and differentiating the first of these equations by y and the second by x and subtracting, it is easily found that

$$\frac{\partial \zeta}{\partial t} = \nu\nabla^2\zeta - u\frac{\partial \zeta}{\partial x} - v\frac{\partial \zeta}{\partial y} \quad \text{.................(39)},$$

where $$2\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \text{twice the spin} \quad \text{............(40)}.$$

Unfortunately these equations in their general form have not yet yielded to solution, except in a number of particular cases and classes of problems. The equations of steady motion in a channel for example can easily be obtained from these by the assumption that v is non-existent and they will be found to coincide with those already obtained. In the same way the problem of steady motion in a pipe may be treated by supposing that the velocities are functions of r only, where $r^2 = x^2 + y^2$, and transforming the equations they will become identical with (30). If the motion be supposed slow so that the terms $u\frac{\partial u}{\partial x}$ etc. can be neglected, the equations are considerably simplified and a large number of problems of this nature can be solved. These include the question of wave motion in a viscous fluid but they are not of immediate practical utility in aeronautics. Recent work by the

authors has shown, however, that the solution of the problem of the steady motion of a body of any shape in a viscous fluid can be reduced to the determination of the flexure of a flat plate loaded and supported in a specified manner, and developments along these lines would seem to indicate a method of attack to which the equations may yield.

A complete solution satisfying the no slip condition at the boundary for some particular problem would be especially valuable for the light it would throw on the problem of how from a state of motion instability and consequently turbulence arises. This demands a mathematical expression showing the origin and history of an eddy, but it has not yet been obtained. The progress of the science is handicapped by insufficient mathematical knowledge.

CHAPTER IV

FROM MODEL TO FULL SCALE

§1. As the result of years of scientific research a vast amount of experimental data has accumulated concerning the lift and drag of various forms of aerofoils, the thrust of propellers and the stability of different types of machines, and the laws that govern these factors for varying forms of design of the corresponding parts. To carry out the greater number of these investigations with full scale machines or parts would not merely be impracticable, inconvenient, and dangerous from the experimental standpoint, but is impossible to any large extent on account of the huge expense that would necessarily be involved. Consequently models approximating only in rare instances to full scale have been the principal objects of study. An attempt to predict the behaviour of a full scale machine or part from the model results brings the designer face to face with a serious problem. A little consideration will soon show that simply to increase lift and drag or propeller thrust of a model in the ratio of linear dimensions of the full scale part to the model, or even according to any simple law that could be framed off-hand, leads to false conclusions. For a complete understanding of the question it is evidently necessary in the first place, therefore, to determine what conditions, if any, allow of the legitimate application of model results and to find the factor and circumstances of transformation. It will be seen that a moderately complete answer to this question may be arrived at by nothing more than a mere application of Newton's Laws of Motion, in their most general form, to the problem under discussion.

It is obvious that the resistance of a body can depend only on its size and shape, its speed, the properties of the fluid through which it moves, and the value of the force of the earth's attraction. The shape may be considered as given by a drawing

of which the length of some part to scale is l, the velocity v, the effect of the earth's attraction conveniently represented by g, the acceleration due to gravity, the density of the fluid ρ, and the coefficient of viscosity μ. As far as the present problem is concerned μ and ρ will be the only constants of the fluid that affect the question, always provided the motion is not so great as to cause the elastic properties of the medium to come into play. Sufficient evidence exists to show that so long as the velocity of the air is restricted below 200 miles per hour it acts as an incompressible fluid.

It follows at once that the resistance R of the body must be uniquely expressible in terms of these quantities, and these quantities only, and it is accordingly legitimate to write

$$R = F(v, l, \rho, \mu) \quad \text{.......................(1).}$$

The precise form in which these four quantities are combined to constitute the function F depends of course on the exact shape of the body, for this is the only factor in the problem that cannot be introduced in terms of a single measurable quantity. In spite of this, however, it is possible to determine quite simply certain general properties of the function F which will yield the desired information. The method that will be adopted is that known generally as the Method of Dimensions and will be explained in detail in the next paragraph.

§ 2. *Dimensional Theory.* Every symbol which can be introduced to describe some dynamical or physical conception such as length l, time t, speed v, acceleration a, force f, etc., must ultimately be expressible in terms of the three fundamental units or dimensions of length, time and mass. A speed, for example, is a distance traversed divided by the time, and consequently if length be represented dimensionally as L, time T, the dimensions of a speed will be L/T. In the same way an acceleration, being a velocity divided by a time, will have dimensions L/T^2, and for a force ML/T^2, since force is proportional to mass multiplied by acceleration in accordance with Newton's Second Law. Under these circumstances the following table may be drawn up:

Quantity	Symbol	Dimension
Length	l	L
Time	t	T
Mass	m	M
Speed	v	$L\ T$
Acceleration	a	$L\ T^2$
Force	f	$ML\ T^2$
Moment	M	$ML^2\ T^2$
Density	ρ	$M\ L^3$
Viscosity	μ	$M\ LT$
Kinematic Viscosity	ν	$L^2\ T$
Constant	c	Non-dimensional

If now for example a force f were to be expressed in terms of any of the other qualities, say v, t, etc., it would follow that these would necessarily be combined in such a manner as to give the dimensions of the total combination the same as that of a force. Quite generally in fact all the terms of any combination of quantities which state some physical or dynamical fact, based on the ordinary laws of motion, must of necessity have the same dimensions. This general property of any combination of such symbols is known as Dimensional Homogeneity. The conception will be made perfectly clear by a particular case. The formula giving the time of oscillation of a simple pendulum is known to be $t = 2\pi\sqrt{l/g}$. The dimensions of the left-hand side being T, that of $\sqrt{l/g}$ must necessarily reduce to the same. The dimensions of $\sqrt{l/g}$ are $\sqrt{(L \div L/T^2)} = T$.

§ **3.** *Application of the principle of Homogeneity of Dimensions.* The idea developed in the preceding paragraph now indicates that a certain degree of restriction must be imposed upon the manner in which the symbols of any equation may be combined. The following examples indicate how this method may be applied.

Example 1. To determine what may be discovered regarding the velocity of gravity waves on the surface of an inviscid incompressible fluid by an application of the foregoing principles.

The velocity of such waves can only depend on the wave length l, the density of the fluid ρ, the gravitational acceleration g. It is accordingly necessary to suppose that these three quantities l, ρ and g are to be combined in such manner that each term in the expression for v has the same dimensions as v.

Suppose a term is quite generally represented by $l^x \rho^y g^z$ so that its dimensions are L M/T^2

$$L^x . (M/L^3)^y (L/T^2)^z = L^{x-3y+z} M^y/T^{2z},$$

which must correspond to L/T the dimensions of a velocity. Equating powers of L, M, T,

$$x - 3y + z = 1, \quad 2z = 1, \quad y = 0,$$

$$\therefore x = \tfrac{1}{2}, \quad y = 0, \quad z = \tfrac{1}{2}.$$

Each term is therefore $\sqrt{lg}$, that is to say there is only one term, and it follows that

$$v = C\sqrt{gl} \quad \text{.............................(2)},$$

where C is some constant; and this is the expression for the velocity of heavy waves on the surface of a fluid.

Example 2. A body starting off with an initial velocity u is under an acceleration g in any direction. It is required to determine what information the preceding theory yields regarding the distance traversed after a time t.

This distance can only depend on the quantities u, g, t and the inclination of the direction of the acceleration at any time to the direction of motion, but the latter cannot be expressed in terms of any definite symbol. Each term in the expression for s, the distance traversed, therefore, written in the form $u^x g^y t^z$ must reduce to the dimensions of s, viz. L. Now the dimensions of $u^x g^y t^z$ are

$$(L/T)^x . (L/T^2)^y . T^z = L^{x+y}\, T^{z-x-2y} = \text{dimensions of } s = L,$$

$$\therefore x + y = 1, \quad z - x - 2y = 0,$$

$$\therefore y = 1 - x, \quad z = 2 - x.$$

$$\therefore u^x g^y t^z = u^x g^{1-x}\, t^{2-x} = g . t^2\, (u/gt)^x.$$

In this case the dimensional theory alone does not furnish sufficient equations to determine x, but it tells us that every term in the expression for s must be represented in the above form, to which so far any value for x must be admissible. s in fact must have the factor gt^2 and the other factor must depend solely on the quantity u/gt. This statement may therefore be written

$$s = gt^2 f\,(u/gt) \quad \text{.........................(3)}.$$

This is all that may be derived from the dimensional theory regarding the distance traversed by a particle under the influence of an acceleration g in any direction. The exact form of the

function f cannot in this case be determined unless the relation of g to the direction of motion etc. at any time be definitely specified. If the acceleration g is always in the same direction as the velocity, it is well known that

$$s = ut + \tfrac{1}{2}gt^2 = gt^2 (\tfrac{1}{2} + u/gt),$$

and therefore in this case

$$f(u/gt) = \tfrac{1}{2} + u/gt.$$

It should now be evident to what this method reduces. The replacing of the units of a force by ML/T^2 etc. is equivalent to Newton's Laws of Motion, and by their application in this very general form sufficient information is derived to limit the form of expression of any quantity, although the exact expression desired will depend on something more. It will depend on the precise manner in which these laws work themselves out in the problem under discussion, and that again will involve such factors, among others, as the shape of the body, etc. It is moreover important to note—and the full significance will appear later—that any formulae we derive will apply so long as the quantities on which the motion has been assumed to depend at any time remain the *only* factors that determine the motion. Should any new factor come into play the whole question will at once be transformed. The fundamental assumption is, that the conditions of the motion, partially expressed by the quantities whose dimensions come into question, are such as to preserve that motion continually consistent with these conditions, and these conditions alone. All the apparent obscurities will be removed from this statement as soon as we come to the problem to which we have gradually led up.

§ **4.** To return now to the real problem of this chapter as introduced in § 1, it was there seen that assuming the elasticity of the fluid does not come into operation to any appreciable extent the resistance of any body immersed in it will be expressible in terms of v, l, ρ, μ, and the shape of the body which, however, cannot be stated or defined by any single symbol. It must be particularly noted that since the time t does not enter explicitly into the question the motion must be such as to involve only the four quantities already mentioned, and the resistance will not vary with time. A steady state of motion has therefore been presupposed. Consider any term in the expression for R of the form,

$v^w l^x \rho^y \mu^z$ whose dimensions must be that of a resistance. The dimensions of this expression are

$$(L/T)^w L^x (M/L^3)^y (M/LT)^z = M^{y+z} L^{w+x-3y-z}/T^{w+z},$$

which must reduce to the dimensions ML/T^2, that of a force.

Hence

$$y + z = 1, \quad w + x - 3y - z = 1, \quad w + z = 2,$$

$$\therefore y = 1 - z, \quad w = 2 - z, \quad x = 2 - z,$$

and each term is of the form

$$v^{2-z} l^{2-z} \rho^{1-z} \mu^z = \rho v^2 l^2 (vl\rho/\mu)^{-z} = \rho v^2 l^2 (vl/\nu)^{-z},$$

where $\nu = \mu/\rho$, the kinematic viscosity. There are apparently not sufficient relations among the quantities to determine z, which may consequently be given any arbitrary value. The most general expression for R will then be represented as a sum of any number of terms of the above form and consequently the most general expression will be given by

$$R = \rho v^2 l^2 [A_1 (vl/\nu)^{n_1} + A_2 (vl/\nu)^{n_2} + \ldots].$$

The terms in the square brackets depending as they do on the quantity vl/ν may be written quite generally $f(vl/\nu)$.

Finally $$R = \rho v^2 l^2 f(vl/\nu) \qquad \text{......................(4)}.$$

§ 5. If a body of given shape be immersed in a viscous fluid and if its resistance be supposed defined purely by the velocity, viscosity, density of the fluid and the size of the body, then the expression for the resistance must take this form. The whole problem of dynamical similarity, and the transition from model to full scale hinges on the discussion of equation (4). It supplies the basis for the experimental investigation by means of models of the aerodynamic problems that occur in aeronautics. The function involved depends for its form merely on the shape of the body as defined by some drawing, but its actual value depends among other things on the size. All bodies of the same shape but of different dimensions immersed in different fluids and moving with different speeds must have their resistances expressible in terms of the same function. In spite of the generality of these results very valuable information can be derived therefrom.

If an experiment be conducted on a model, a length of one part of which is l, in fluid of viscosity ν and density ρ moving with

velocity v, and another on the corresponding full scale body of length l, in another fluid of viscosity ν_1 and density ρ_1 and velocity v_1, then resistance of model $= R = \rho v^2 l^2 f(vl/\nu)$,

and resistance of full scale $= R_1 = \rho_1 v_1^2 l_1^2 f(v_1 l_1/\nu_1)$.

If the velocities, viscosities, and densities, be so selected that

$$vl/\nu = v_1 l_1/\nu_1 = a,$$

then

$$R'/R_1' = \rho a^2 \nu^2 f(a)/\rho_1 a^2 \nu_1^2 f(a) = \rho\nu^2/\rho_1\nu_1^2 \quad \text{......(5).}$$

The resistance of the full scale is thus expressed in terms of that of the model, where it is especially to be noted that the resistance of the model R is that at velocity v, and that of the full scale R_1 at velocity $v_1 = vl\nu_1/l_1\nu$. v and v_1 are defined as corresponding speeds for bodies of the same shape whose dimensions are given by l and l_1 immersed in fluids of kinematic viscosities ν and ν_1 respectively.

Before proceeding further it may be necessary to restate some of the conditions under which equation (5) is applicable as assumed in its derivation and the implications that these involve. It was in the first place supposed that the motion can be completely expressed in terms of ρ, v, l, ν, and consequently does not depend on time explicitly. It follows that a picture of the flow and stream lines at any time will always represent the flow and stream lines. It is geometrically fixed and the motion is steady. Since

$$vl/\nu = v_1 l_1/\nu_1,$$

and this must be supposed to hold for all corresponding points in the two fluids, the velocities at these points bear a fixed ratio to each other depending merely on the nature of the fluids and on the relative sizes of the two bodies. The expression deduced for R must also be the form in which all the forces and their components (shearing, etc.) which exist at corresponding points can be thrown, and these also bear a constant ratio to each other for corresponding points depending merely again on the nature of the fluid and size of the bodies. All the forces being thus equal in magnitude to a constant factor, and the same in direction, the state of flow round the two bodies and the shape of the stream lines must also be identical. Photographs of the two states of steady motion past the bodies taken on plates of the same size would be exactly the same, and the two motions are said to be *dynamically similar*. Fig. 48 furnishes a striking comparison

COMPARISON BETWEEN FLOW IN WATER AND IN AIR

Low value of $v l$ Water

High value of $v l$ Water

High value of $v l$. Water

Fig. 48 *a*

Low value of vl. Air

High value of vl. Air

High value of vl. Air

FIG. 48 *b*

between the nature of the flow in air and that in water where the similarity is well brought out, considering the experimental difficulties involved.

§ 6. It is now evident that vl/ν is a most important quantity in any physical experiment on fluid motion, and provides the criterion that the experimental facts derived from a model be directly applicable to the corresponding full scale equivalent. In general, however, conditions are from one point of view slightly more simple, the model experiment and the full scale performance taking place in the same medium, viz. air. Accordingly $\nu = \nu_1$ and the criterion obtained demands

$$vl = v_1 l_1 \quad \text{..........................(6),}$$

the densities and viscosities in the two cases being the same. Equation (5) now becomes

$$R'/R_1' = \rho\nu^2/\rho_1\nu_1^2 = 1 \quad \text{..................(7).}$$

When the experiment on the model is carried out at the same value of vl as for the full scale, the resistances experienced in the two cases are equal. It is this last deduction from the theory of dynamical similarity that is of the most frequent and direct application, but it demands a state of affairs which is extremely difficult to attain in the course of normal experimental work. For models of aerofoils tested in the artificial winds in channels at aerodynamical laboratories, it is usually convenient to deal with models about 1/25 full scale, and consequently to attain values of vl of the order required for a full scale part it would be necessary to carry out experiments at 25 times the velocity of flight of the machine, or roughly 2000 m.p.h. To obtain anything even approaching this is manifestly impossible. The highest velocity attainable in the new 7 ft. channel at the N.P.L. does not exceed 60 m.p.h. Moreover even if the full vl could be reached the forces which are brought into play, being equal to those of the full scale machine, would be too great to deal with, and measure accurately. It would appear therefore at the very outset that in this respect model experiments must be hopelessly inadequate for full scale predictions, but a practical method has been developed which works itself out very satisfactorily. By determining the resistance of a model for the highest range of speeds possible and dividing by $\rho v^2 l^2$ the value of the function

$f(vl/\nu)$ is determined for each value of vl/ν in that range. If these be plotted against vl/ν, as will be done several times in Chap. VI, it is found that as vl/ν increases the curve tends to rise asymptotically to a certain definite value. This is equivalent to the statement that the resistance tends to increase steadily as the square of the velocity, for large values of vl/ν.

Consequently what is normally done is, if possible, to carry through the experiments up to such values of vl/ν, or what amounts to the same thing in this case vl, as show the value of $R/\rho v^2 l^2$ to become sensibly constant, and then to extrapolate by extending this curve to the higher values of vl. In practice for ease in extrapolation it is customary to plot horizontally not vl but $\log vl$ for this increases the horizontal distance for which we know the values of $R/\rho v^2 l^2$ and decreases the extrapolated range. Many of the figures occurring in Chap. VI illustrate this method. A complete discussion, with applications, showing how to pass from experimental results on model aerofoils to the corresponding full scale will be taken up in a later chapter (Chap. VI), but the following examples will clear the ground as far as is necessary for the present.

The following definitions of coefficients will facilitate further discussion.

K_d = *Drag Coefficient* = Component D of resultant wind force parallel to wind divided by $\rho l^2 v^2$.

Hence $K_d = D/\rho l^2 v^2$.

K_l = *Lift Coefficient* = Component L of resultant wind force perpendicular to wind direction divided by $\rho l^2 v^2$.

Hence $K_l = L/\rho l^2 v^2$.

In the case of an aerofoil l^2 is taken as the area of the plane.

The coefficients K_d and K_l are clearly non-dimensional and their values are consequently absolutely independent of the system of units adopted. In this respect they are uniquely suitable as a universal means of representing the aerodynamic properties of the bodies.

Example 1. The following table gives the resistance coefficients of aeroplane stream line wires, of fineness ratio 3 : 1, the model tested being 1″ × 3″, and 20″ long, at given wind speeds. It is required to determine the resistance of a wire of the same cross-section, breadth 1/3 inch, at a wind speed of 150 ft./sec.

Velocity (ft./sec.)	Value of vl (ft./sec. units)	Resistance per foot run (lbs.)
20	1·67	0·0063
25	2·08	0·0099
30	2·50	0·0112
35	2·92	0·0194
40	3·33	0·0252
45	3·76	0·0320
50	4·16	0·0395
55	4·58	0·0477
60	5·00	0·0570

Value of vl for given wire at given speed equals 50/12. Hence resistance from table = 0·0395 lb. per ft.

Example 2. Fig. 74, Chapter VI, shows the variation of L/D for an aerofoil with log vl for various angles of attack (l = length of chord). The experiments from which these were derived were conducted on a plane of dimensions 15″ by $2\frac{1}{2}$″ for a range of speeds 10 ft./sec. to 50 ft./sec. The curves have been extrapolated for higher values of vl as shown. It is required to determine the value of L/D for a plane of the same shape but dimensions 30 ft. by 5 ft. when the wind speed is 100 ft./sec. and the angle of attack 2°.

Value of vl for full scale $= 100 \times 5 = 500$. Hence $\log vl = 2{\cdot}7$. From fig. 74, L/D for this value of $\log vl$ is 19.

§ **7.** The method that has been outlined in finding a form for the resisting forces in a fluid can similarly be developed for the moments of the forces which are brought into play. The student will easily prove, on the assumptions already made, that M the moment on the model is given by

$$M = \rho v^2 l^3 f(vl/\nu) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8).$$

The calculations for the passage to the full scale are carried out as already indicated, but it may be noted that, unlike the resistance, moments at the same vl are different.

§ **8.** *Ship Models.* The principle of dynamical similarity is not restricted in its application merely to aerodynamic questions, but is likewise of extreme importance in the corresponding problems that arise in navigation, and experiments on model ships. It does not now suffice to say that the resistance will depend only on the

quantities ρ, v, l, and ν. The energy lost in wave-making makes its appearance in the form of an additional resistance, and this must now also be taken into account. Assuming that the waves produced are gravity waves this extra factor may conveniently be represented by introducing the acceleration due to gravity g. The resistance involving terms due both to skin friction and wave-formation will, accordingly, depend on ρ, v, l, ν, g, and it will be justifiable to write each term entering into the expression for R in the form of $\rho^p v^q l^r \nu^s g^t$. The dimensions of this quantity must of necessity reduce to that of a force. Proceeding as before

$$(M/L^3)^p\,(L/T)^q\,L^r\,(L^2/T)^s\,(L/T^2)^t = M^p\,L^{q+r+2s-3p+t}/T^{q+s+2t}.$$

Equating this to ML/T^2, we find

$$t+q+r+2s-3p=1, \quad p=1, \quad q+s+2t=2,$$

$$\therefore\ q=2-s-2t, \quad p=1, \quad r=2-s+t.$$

Hence the terms become

$$\rho v^2 l^2 . (vl/\nu)^{-s}\,(gl/v^2)^t.$$

Since any value may apparently be given to s and t without violating the conditions that homogeneity of dimensions demand, it will be justifiable to write

$$R=\Sigma A\rho v^2 l^2\,(vl/\nu)^m\,(gl/v^2)^n,$$

or as before $$R=\rho v^2 . l^2 . f\,(vl/\nu,\, gl/v^2).$$

The term vl/ν accordingly as before is the factor that introduces the skin friction, while gl/v^2 corresponds to the term for the wave-making resistance. Following the lines developed in the simpler case where this latter component of the force is assumed non-existent, the form of the function f will depend in some manner on the shape of the model, and is, as far as we are concerned, not so far determinable. If, moreover, it is attempted to avoid this difficulty after the manner adopted in the earlier examples, it would be necessary to carry out the model and full scale experiments in such a manner as to keep both vl/ν and gl/v^2 constant. Using the same notation as before it follows that

$$vl/\nu=v_1 l_1/\nu_1 \text{ and } gl/v^2=g_1 l_1/v_1^2.$$

It is in general not possible to arrange that the experiment on the model be carried out in a fluid of different viscosity from

that of the full scale (viz. water), and consequently ν is in general equal to ν_1. The conditions then become

$$vl = v_1 l_1 \text{ and } l/v^2 = l_1/v_1^2,$$

which is at once seen to involve

$$l = l_1, \quad v = v_1.$$

The experiments on the model and on the full scale are identical, and the method apparently fails.

To overcome this difficulty certain assumptions are made which, if not rigorously correct, are at least sufficiently close to the truth to give accurate enough results and deductions upon which to base calculations. It is assumed that the resistance due to the skin friction is quite independent of that due to wave-making and would exist whether waves be formed or not. Further the wave-formation is itself supposed quite independent of viscosity. It follows at once that the expression for R must necessarily be separable into two portions corresponding to each of the two separate components of the resistance, the one involving vl/ν and the other gl/v^2.

Hence $\quad R = \rho v^2 . l^2 . f(vl/\nu) + \rho v^2 . l^2 . F(gl/v^2).$

The first term in the expression is the frictional resistance and the second portion obeys Froude's Law of corresponding speeds, where these are such as to keep gl/v^2 constant. Both parts of the expression for R may now be determined separately by experiment.

§ **9.** In § 3 it was definitely stated that so long as the conditions of any system of motion are such as to maintain continually the truth of the initial assumption that ρ, v, l, ν are the only quantities on which the whole motion depends, so long will the formula there deduced from the principle of homogeneity of dimensions apply. The implication that would seem to be contained in this statement is apparently that the motion is steady, but, although so far the principle of dynamical similarity here elaborated has been restricted to the case of steady motion, this was quite unnecessary. A much wider conception of this principle may be developed. It is proposed therefore to extend the conception at once to those cases in which the motions at every point of the fluid undergo a series of transformations, so that the motion is not steady in the mathematical sense. It follows at once that

we can no longer state as formerly that such quantities as the resistance will depend merely on the values of the quantities ρ, v, l, ν, but must now of necessity involve also the time t. If it be assumed that the motions are periodic, that is to say, they repeat themselves at regular intervals, it is obvious that the period of this motion can as before depend only on the four quantities ρ, v, l, ν, and the shape of the body. If t be this period it must be possible to express it in the form

$$\rho^x v^y l^z \nu^n.$$

This must reduce to the dimensions of t. Accordingly following the method already applied it is easily found that

$$t = lf(vl/\nu)/v \qquad (9).$$

Comparing the period of the state of flow that results round two bodies of the same shape, but of different sizes, when moving through fluids of different viscosities but at corresponding speeds where vl/ν is the same for both, it is easily seen that

$$t/t_1 = lv_1/l_1v = (v_1^2/v^2) = (l^2/l_1^2),$$

assuming for the moment that the viscosities are equal.

Accordingly the periods of the motions when at the same vl are directly as the squares of the corresponding sizes of the bodies. Interpreting these results physically it is quite evident what is taking place. The series of transformations through which the fluids pass are identical but they take place at different rates. The larger the body the greater is the period of the flow and therefore the slower is the motion. If a series of photographs of the two systems of flow were taken on plates of the same size at corresponding times as given by the expression here found for the period, the two series would be identical. If for example eddies were being formed and thrown off periodically behind a body in a fluid of given viscosity, eddies would similarly be set up in a fluid of different viscosity round a body of the same shape and moving at the corresponding speed, but the rate of their formation, etc. would be different. This conception of the mechanism of fluid motion gives the real significance of the principle of dynamical similarity in its widest sense.

§ **10.** It must not be supposed that this limited application of the results of experiments on models to full scale is the only direction in which model results can be utilised. One of the

most important applications in fact of such research finds its place in the question of design. In the foregoing discussions it has always been assumed that the shape of the parts has already been determined of such form as to be most suitable for their respective functions. But these have not been discovered without an elaborate series of experimental comparisons. What is done, for example, in the case of aerofoils is that a series of experiments are conducted in the wind channel to test the lift and drag for different forms of sections. Results so obtained relating simply to shape are generally applicable directly to full scale and it is simply in the quantitative deductions in passing for the forces from the model to those on the full scale that the *vl* effect has to be considered. It is thus in the improvement of design by a comparison of the relative effects of modifications in the shape of parts that model experiments finds one of its most fruitful applications.

PART II

CHAPTER V

THE AEROFOIL

CHARACTERISTICS REQUIRED FOR SPECIAL TYPES OF MACHINES

§ **1.** *Introduction.* In the preceding chapter the method of transition from results obtained by experimental investigation on models to full scale aerofoils was fully discussed, and it is now evident that the direction in which the investigation must be developed will be determined by the functions the aerofoil is required to fulfil. Variations in camber both on the upper and lower surface, aspect ratio, plan form, form of entry and trailing edge all have their special significance and these show themselves in particular, in the values of the Lift and Drag coefficients and the ratio L/D. Working conditions, differing for each type of machine according to the requirements, will determine upon which of these particular coefficients or combinations of them special emphasis should be laid. An indication of a general nature of the special type of aerofoil required for some purpose can often be obtained from an analysis of the corresponding form where it occurs in nature. Some useful information has been derived from the study of birds' wings.

§ **2.** It is evident that the main characteristics which determine the performance of an aerofoil are, stated generally,

(1) The shape of sections,

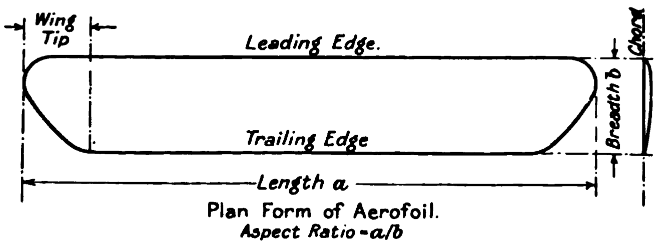
(2) The aspect ratio, i.e. total length divided by width,

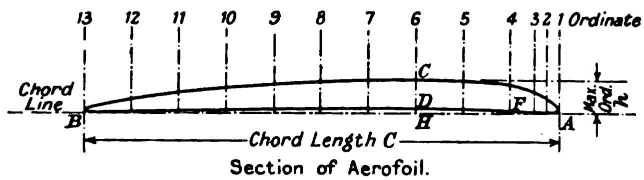
(3) Plan form (see fig. 49),

and in order to attain to the type of wing required for any special purpose it is necessary to consider in detail the effect of variations in each of these. This it is proposed to do in Chapter VI.

All the properties of the aerofoil that will affect the actual performance as far as we are concerned at present will be con-

tained in curves giving the variation of K_l and K_d, the lift and drag coefficients (Chapter IV, § 6), with the angle of attack α, but it will be found more convenient to discuss the question by making use of the equivalent curves as in fig. 50, where K_l is plotted against α, and L/D against K_l.





R.A.F. 6

Ordinate	Distance from leading edge in terms of chord	Height above chord in terms of chord	
		Upper surface	Lower surface
1	0·0	0·005	0·000
2	0·025	0·032	—
3	0·05	0·044	0·002
4	0·1	0·060	0·004
5	0·2	0·074	0·007
6	0·3	0·076	0·008
7	0·4	0·075	0·007
8	0·5	0·071	0·005
9	0·6	0·065	0·004
10	0·7	0·057	0·003
11	0·8	0·044	0·002
12	0·9	0·027	0·001
13	1·0	0·005	0·000

FIG. 49

A the point of no lift usually occurs at a negative angle of attack of about $-2°$ or $-3°$, i.e. the chord is slightly inclined downwards to the direction of the wind. The point of maximum

lift available occurs at C and the corresponding angle of inclination in that position is called the critical angle, usually 14° or 15°, at which either an increase or decrease in the angle of attack would cause a decrease in the lift. From A to C the lift increases steadily, the lift curve in general being practically straight. B at which the maximum value of L/D is obtained is of special importance and usually occurs at about 3° angle of attack.

Since the wing is the principal factor in providing the lifting force upon the machine, the vertical component of the resultant thrust upon it must be equal to W the total weight of the aeroplane when it is in steady horizontal flight, and accordingly

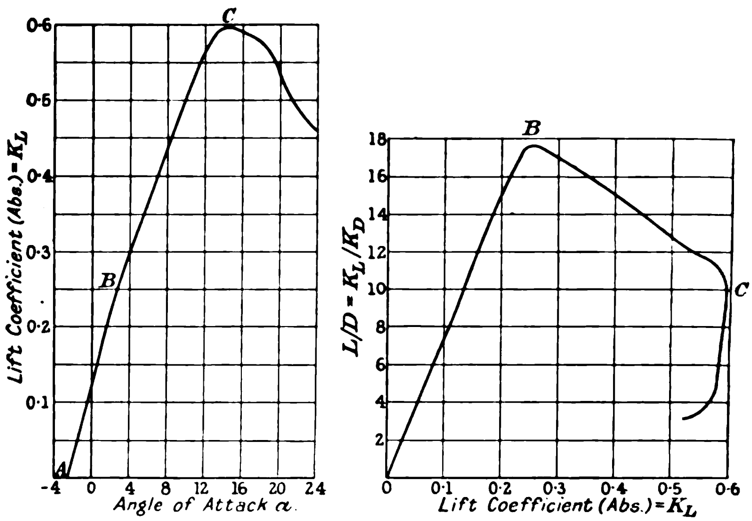


FIG. 50

$W = \rho A V^2 K_l$, where K_l is the lift coefficient for the wing at that angle of incidence. Hence

$$V = \sqrt{(W/\rho A K_l)} \quad \text{.....................(1).}$$

Corresponding to every value of K_l this equation determines the speed at which the machine must fly to maintain horizontal flight, and if the angle of attack be increased so that K_l increases as between A and C in fig. 50, this speed will diminish, but at C where K_l has reached its maximum value the speed of flight will have reached its minimum, i.e.

$$V_{\text{min.}} = \sqrt{[W/\rho A \, (K_l)_{\text{max.}}]} \quad \text{..............(2).}$$

A simple consideration will show that this minimum speed should be the landing speed S, because when it is desired to land a machine, it is evidently advisable in the first place that there should be no shock or impact, and consequently the machine must be flying practically horizontal on touching the ground. In the second place, external conditions such as the state of the ground or the near presence of obstacles necessitate landing at the slowest possible speed. Landing, therefore, ought to be executed at the minimum speed of the horizontal flight. The normal landing speed for present-day machines lies in the neighbourhood of 50 to 60 m.p.h. For a machine of given weight the greater the area of the wings the smaller S will be. Moreover for normal flight the load per unit area,

$$P = W/A = \rho V^2 K_l \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

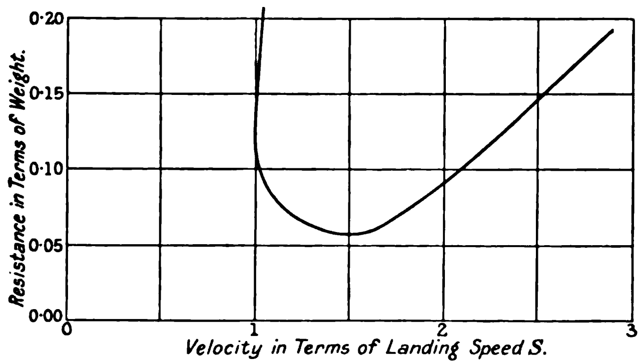


FIG. 51

will be a measure of the landing speed for

$$S = \sqrt{[P/\rho\,(K_l)_{\text{max.}}]} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(4),$$

and the greater P is the greater must S be. P usually ranges between 5 lbs./sq. ft. and 9 lbs./sq. ft.

Using equations (1) and (2) it is easily seen that

$$V = S\sqrt{[(K_l)_{\text{max.}}/K_l]} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(5),$$

giving an expression for the velocity of the machine in terms of the landing speed for any angle of attack, where the value of K_l at that angle of attack is found from the curve in fig. 50. From equation (5) a new curve may be derived with which it will be very convenient to work. Assuming a value for the landing speed S, V can be obtained at any value of the angle of attack α by simply

taking $(K_l)_{max.}$ and K_l from the lift curve. Assuming the weight of the machine which of course equals L the total lift, the value of D, the total drag on the wings for this angle of attack, can be derived from the curve giving the variation of L/D with α and consequently the variation of D with V can be plotted as shown in fig. 51.

The resistance offered by the aerofoil itself as distinct from the remainder of the machine does not in general exceed more than a third of the total drag. Since this head resistance has to be overcome by the propellers through the medium of the engine, it follows that the greater the head resistance the greater must be the engine weight and petrol consumption. Accordingly a first principle in design must involve attempts to reduce this resistance to a minimum by the shaping of all parts of the machine which come into contact with the wind. For our purpose it will be convenient to separate these resisting forces into two parts; firstly that which is offered by the main body of the machine and the various parts necessary in the construction distinct from the aerofoil itself, and secondly by the aerofoil. The first set offers to motion a resistance varying little with the attitude of the machine and approximately proportional to the square of the velocity. As far as variation in the type of aerofoil is concerned this is a head resistance which becomes of considerable magnitude at high speeds. For any machine, therefore, no matter what its function is intended to be, it is always necessary in the first place to reduce the resistance by fairing and shaping the exposed parts as explained in Chapter II consistent with other requirements. For convenience in discussing the type of aerofoil required for some special purpose it will be necessary therefore to suppose that the general body of the machine and its weight have been determined, and to consider only the effect of such variations as camber, entry, etc. on the aerofoil itself.

§ **3.** *High Speed Machines.* Consider in detail the characteristics an aerofoil must possess designed for a high speed machine. The factor that will tend to operate against the attainment of high speed will evidently be the resistance of the aerofoil that is brought into play by the motion, assuming that the remaining parts of the machine as already explained have been determined upon. The fact that the total weight of the machine

is given is equivalent to the statement that the aerofoil must have a lifting force exerted upon it equal to this given weight. Accordingly the aerofoil would require to be so designed that at the high speeds of flight contemplated the resistance must be a minimum, or, since the lift L is given, being equal to the total weight of the machine, the L/D of the aerofoil at the contemplated speed of flight must be more than that of any otherwise suitable aerofoil. It should be specially noted that other aerofoils may exist having a greater value of L/D than the selected aerofoil, but they would not necessarily satisfy the remaining requirements

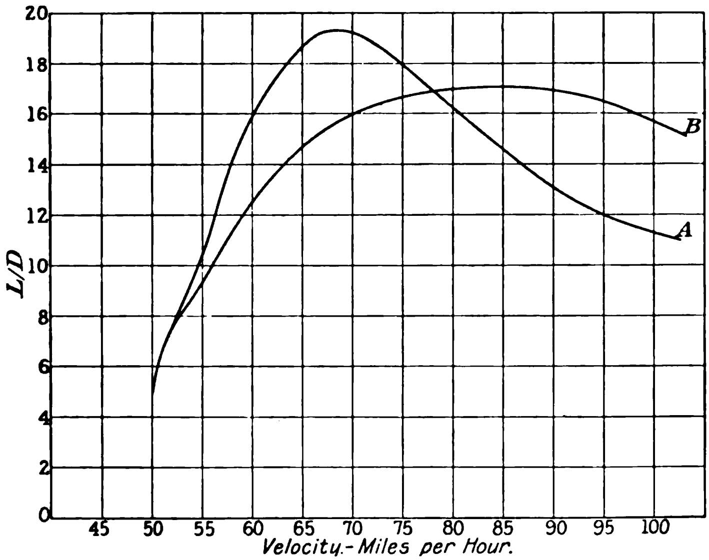


FIG. 52

such as for example possessing a presupposed value for S, the landing speed, the minimum speed of flight. The following comparison between aerofoils will bring out these points.

The two examples are taken from results published by Eiffel in *La Résistance de l'Air et l'Aviation*. For comparison purposes the landing speed must be selected as equal in both cases, $S = 50$ m.p.h. say. From the published results upon two aerofoils A and B, given as No. 35 and No. 37 in the work referred to, the L/D ratio has been plotted against the velocity of the machine in fig. 52, using equation (5).

If V is the speed of flight, taken as 100 m.p.h., it is easily seen that L/D for B at that speed is greater than for A, and accordingly B is a much more suitable aerofoil for the present purpose than A.

Taking the resistance of the rest of the machine as 250 lbs. at 100 m.p.h., the weight (equal to L) being 2000 lbs., the drag of the wings for A would be 177 lbs. and that for B 131 lbs., and therefore if the machine was constructed with wings A the total resistance would be 427 lbs. and if with B, 381 lbs. The horse-power required to overcome this is 114 in the one case and 101 in the other, so that in order to carry sufficient power to work at the same speed a greater part of the useful weight in the case of A would require to be sacrificed, and consequently if the same weight was devoted to the engine in both cases or if both engines were constructed to give the same horse-power B could fly at a much higher speed than A. It should be remarked in conclusion that for an aerofoil satisfying these requirements the greatest

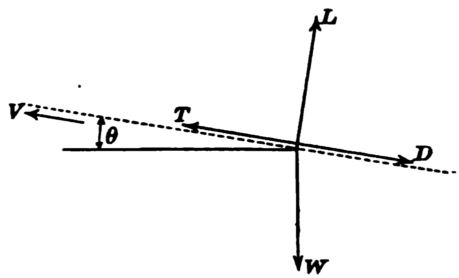


FIG. 53

value of its L/D usually occurs at a speed considerably lower than the high speed required, but this is not really a disadvantage since flight must take place in the range S to V.

Summing up, therefore, the best type of aerofoil for a high speed machine is one having the highest L/D at lift coefficient given by $(K_l)_{\max.}\, S^2/V^2$. For modern machines this occurs at low values of K_l of approximately 0·1.

§ 4. *Climbing Machines.* As in the previous section, the aerofoil is not the only factor which enters into the problem of efficient design for climbing. The engine and propeller for example bear an important relation to this question, but as far as the present section is concerned, it will at the outset be supposed that

these have been predetermined and the machine may be assumed already constructed most efficiently for climbing except for the aerofoil.

If V be the speed of the machine along its flight path which at any moment is inclined at an angle θ to the horizon, fig. 53, then the rate of vertical climb is $V \sin \theta$, being the distance moved per second vertically. It is evident that if both V and θ could be suddenly increased the rate of climb at the instant at which this occurred might reach a considerable value, but this is not what is generally intended by the expression when referring to good climbing machines. Any aeroplane, no matter what its function, by a suitable combination of circumstances might climb very rapidly for a short time in a spurt, but a good climbing machine is one whose rate of climb can be kept steadily at a high value for a considerable period. The criterion therefore is that $V \sin \theta$ must be a maximum when V is the speed of flight along the path.

Resolving the forces on the aeroplane along and perpendicular to the flight path,

$L = W \cos \theta = \rho A K_l V^2$ ....(6),

$T = D + W \sin \theta$, since it is assumed there is no acceleration, where W = weight of machine..............................(7),

L = total lift on the wing perpendicular to the path,

D = the drag on the machine,

T = the propeller thrust assumed acting along the flight path.

Equation (6) determines the velocity in the flight path for various values of K_l. In general the angle of best climb may be approximately 8° to 9°, so that the error involved in V^2 in taking $\cos \theta = 1$ is less than 1 per cent. Since $V \sin \theta$ is the rate of vertical climb, $WV \sin \theta$ is the rate at which work is being done against gravity, but using (7),

$$WV \sin \theta = (T - D)\, V \quad \text{.................(8)},$$

and accordingly since $V \sin \theta$ must be a maximum for the best rate of climb, $TV - DV$ must be a maximum. TV is the power supplied by the engine and propeller, and DV is that portion of it used up in overcoming the head resistance of the machine. The criterion for maximum rate of climb demands that the difference between these be a maximum.

The head resistance is composed of two factors, the body resistance of the machine and the drag on the aerofoil. Fig. 54 gives two typical curves corresponding to each of these parts. By adding their ordinates and multiplying by V at each point the curve giving the variation in H.P. ($= DV$) required to overcome this head resistance is found. Fig. 54, curve b, is such a curve. A series of curves giving the variation of the available H.P. (TV) with speed may evidently be obtained corresponding to various degrees of engine throttling, but for our purpose it is evidently necessary to take that one of the series which corresponds to the

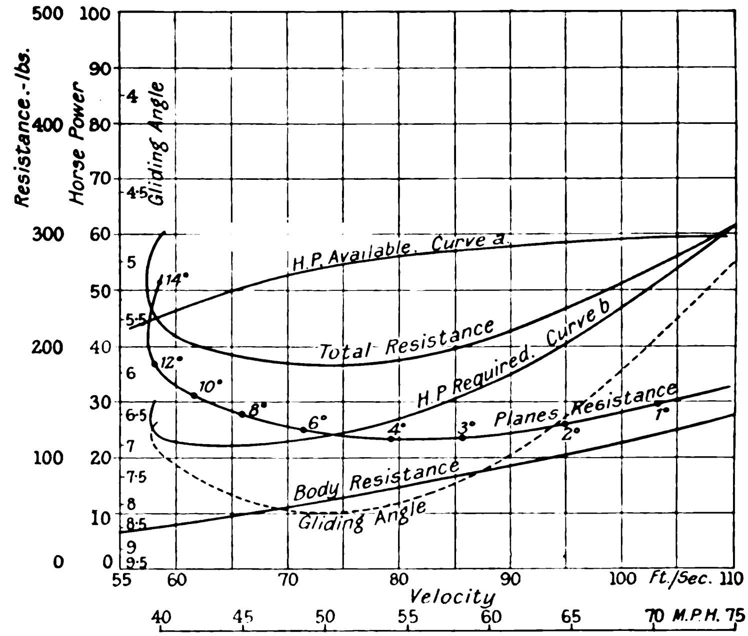


FIG. 54

maximum H.P. that the engine and propeller can give. Curve a, fig. 54, gives a typical curve corresponding to the available H.P. that may be derived from the engine and propeller. The difference between the ordinates of the two curves a and b divided by W, the weight of the machine, is equal to the greatest rate of climb at that speed.

The maximum climbing speed will be given by the horizontal coordinate corresponding to the position at which the distance

measured vertically between the two curves *a* and *b* has reached its greatest value. This evidently occurs in the region of minimum drag or, since the lift is here practically determined for the aerofoil by the total weight of the machine, the maximum L/D that can possibly be obtained from the aerofoil.

The distinction between the conclusion here arrived at and that of the previous section where high speed machines were discussed must be kept clear. What was discussed there was the highest L/D that could be obtained at a predetermined speed, which would not necessarily be the speed corresponding to the maximum value of L/D for the selected aerofoil. Here, on the other hand, the flight will take place in the neighbourhood of the speed which gives maximum L/D for that aerofoil. It is of course clear that the conclusions here arrived at cannot be regarded in any sense as being strictly true but must be taken merely as indications in what direction to look for guidance. As already pointed out other considerations such as propeller and engine would require to be taken into account for a full and complete discussion. As a general rule however that aerofoil among a series most suitable for a climbing machine will be the one whose $(L/D)_{\text{max.}}$ is the greatest of the series. For example aerofoil No. 2, § 3, Chapter VI, has a value of maximum L/D equal to 14·85, large compared to the maxima of all the others, and accordingly this may be regarded generally as suitable for a good climbing machine. The final choice would depend on other considerations.

§ **5.** *Weight Carrying.* It is perfectly evident at the outset that any weight saved in actual construction of the machine itself that would not at the same time involve a decrease in strength of build may be regarded as so much weight saved for useful carrying. Assuming the machine has already been built to satisfy these requirements, it is proposed now to consider from the point of view of weight carrying what effect alterations in the aerofoil, its size and shape, may make. The two factors which may be considered as most important are

(*a*) increase in wing area,

(*b*) alteration in the section of the aerofoil.

By increasing the area of the wings it is evident that a greater lifting force will be brought into play, but at the same time extra

weight of material must consequently be devoted to the constructional parts of the machine in order that such increase should not militate against its strength. Moreover, the extra drag which is of necessity involved by such increases would require to be compensated for, by an equivalent increase in H.P. and therefore weight of engine and fuel. An increase in wing area brings into operation two continuously conflicting factors, an increase in lift and an increase in dead weight. The limit of design will have been reached for our present purpose when the difference between the total lift obtained from the wing and the total dead weight of the machine required for that wing is a maximum. To all intents and purposes the total lift is directly proportional to the wing area, but it is evident that no such simple law other than empiric could possibly be framed for the variation in weight of constructional parts (including engine and fuel) with wing area. Practical experience alone must decide this point. To consider the effect of (*b*) it will be convenient to assume that the weight W of the machine and the landing speed S are fixed.

Compare the effect of successively fitting two aerofoils of different sections to the machine. In order that the total weight may be unchanged they must of necessity have different wing areas, for if K and K' be the maximum lift coefficients

$$W = \rho AKS^2$$

and $W = \rho A'K'S^2$, where A and A' are the wing areas.

Hence $A'/A = K/K'$ so that the difference between the wing areas

$$A - A' = A\,(1 - K/K') \quad \text{.....................(9).}$$

This is the wing area saved by taking the aerofoil whose maximum lift coefficient is K' instead of K and therefore it gives a measure of the extra weight saved for useful carrying. Assuming the machine already in the neighbourhood of the limit of design contemplated, K does not differ greatly from K' and this extra weight is approximately

$$\rho S^2 K\,(A - A') = \rho S^2 A\,(1 - K/K')\,K$$
$$= W\,(1 - K/K') \quad \text{..............(10).}$$

If K be the highest maximum lift coefficient of a series of aerofoils of which K' is the maximum lift coefficient of one of them, evidently by substituting K for K', weight will be gained and

accordingly the criterion arrived at from this consideration is that the aerofoil best suited for weight lifting is the one of a series which has the highest maximum lift coefficient.

Regarding the question from another point of view it will be seen that there is a second factor in the design of an aerofoil for the present purpose which must be considered. If V ft./sec. be the flying speed and η the efficiency (L/D) of the machine, then $WV/550\eta$ will be the B.H.P. of the engine and propellers to keep up the flying speed, where W is in lbs. The weight of the engine and fuel, taken as proportional to this, may therefore be written

$$a \,.\, WV/550\eta = W/\lambda \text{ say} \quad \text{.................}(11).$$

If η' be the efficiency of the machine when another aerofoil has been substituted for the first, its wing area, etc. being changed to keep the other factors constant, then the weight of the engine and fuel will be $aWV/550\eta'$ and the amount saved that may be used for useful carrying is

$$aWV\,(1/\eta - 1/\eta')/550 = W\,(1 - \eta/\eta')/\lambda \quad \text{.........}(12).$$

As before the limit of design as far as variation in efficiency only is concerned will be reached when the above expression has attained its maximum value. Assuming as has been done that the machine itself is otherwise fixed and only variations considered in its maximum efficiency caused by variations in aerofoil, this equation demands maximum efficiency for the latter. It is possible that these conclusions might not always be completely consistent with the results already obtained, namely that the highest maximum lift coefficient is desirable, but an elementary consideration will show which of these two factors is the more important. The total useful load saved by variation both in maximum lift coefficient and efficiency is

$$W\,(1 - K/K') + W\,(1 - \eta/\eta')/\lambda,$$

and the effect of the term giving the saving in load by taking the aerofoil of highest efficiency is only $1/\lambda$ times the saving obtained by taking the highest maximum lift coefficient. In practice λ is approximately 7 so that the effect of a percentage change in maximum lift coefficient will be 7 times that of the corresponding percentage change in efficiency of the machine. At the same time the effect of any change in the efficiency of the aerofoil can only be considered in so far as it produces a change

in the efficiency of the total machine and would require, therefore, to be of considerable magnitude in order to be at all comparable to a change in the maximum lift coefficient. Summing up, therefore, it may be stated that the paramount consideration for weight carrying machines is to select from a series an aerofoil which has the maximum lift coefficient, but if two such aerofoils are practically equal in this respect that one having the greatest *L/D* for the required speed is the better. These considerations are totally apart from any effects of change in construction or build of the aeroplane. It is evidently possible for example to increase the total lift by multiplying the number of planes, but this falls naturally for consideration under (*a*) where the effect of wing area was considered.

§ **6.** The most important classes of machines that are met with in practice have been enumerated and described, but it must be remembered that an aeroplane is very rarely constructed purely for one purpose and accordingly every machine is in the nature of a compromise. This is seen especially to be the case when it is remembered that many other factors may enter that have not been considered. In particular variations in the density ρ of the atmosphere may have a considerable effect on the efficiency of the machine for climbing purposes. The table in Chapter II gives these variations with height and a little consideration shows that this change reacts at once on such questions as climbing power. The lowering of the pressure of the atmosphere causes an insufficient supply of air to the engine. At the same time the propeller must run very much faster since the machine is to have its air speed considerably augmented in order to obtain sufficient lift when the density of the air falls. Apart from the considerations of the effect of cold, and diminished pressure, upon the pilot these factors operate to set a limit to the height to which a machine may rise. A machine best suited for flying at high altitudes will evidently be a combination of a good climbing machine as far as the aerofoil is concerned, and a high speed machine from the point of view of the propeller and engine.

The principles to be applied in the design of a complete machine are given in Chapter XIII, but the general lines of argument followed in the discussion of the three types considered here may be regarded as representative of the method to be

adopted in the selection of the type of aerofoil for a given purpose. They indicate the characteristics that must be sought. In the next chapter it is proposed to enter fully into the experimental investigations that have been undertaken and the results obtained with different aerofoils. A thorough knowledge of this will be of primary importance as an aid to design.

CHAPTER VI

THE AEROFOIL (*cont.*)

Experimental Investigations

§ **1.** The effects of modifications in shape, etc. produced on the forces that are brought into play on the aerofoil have been the subject of more or less systematic investigations at the few aerodynamical laboratories at present existing. The experiments are conducted in the wind channels upon models approximately $\frac{1}{15}$ to $\frac{1}{20}$ full size, and the values obtained for the forces must be modified for scale effect, according to the principles outlined in Chapter IV and developed later in this chapter, before being applied to a complete machine. As far as comparative results between aerofoils are concerned, however, the values measured in the wind channel may be directly used. In the pursuit of such research the main object that must be borne in mind is that the effect of any given change should become clear in itself, and not confused with those due to other variations, but this is of course extremely difficult to realise in practice.

As already stated in the previous chapter the performance of wings may be varied by any of the following changes:

1. Altering the shape of the section.
2. Altering aspect ratio.
3. Altering the plan form.

And almost all the characteristics that are in practice desired for an aerofoil can be investigated under these headings. They will be discussed in the order given.

1. *Alteration in shape of section.*

§ **2.** The cross-section of an aerofoil as usually presented to the draughtsman is shown in fig. 49, Chapter V, where the outline

of the lower and upper surfaces of the wing is determined by the coordinates of points on them expressed in terms of the chord AB.

Whenever anything critical as regards the effect of the shape occurs, it is necessary that a larger number of measurements in that region be given. A reference to the figure will show that the more important characteristics of the outline are practically determined by the position and magnitude of the maximum ordinate of the upper surface at C, the form of entry or leading edge CAF, and the position and magnitude of the maximum ordinate D on the lower surface. The ratio CH to AB is called the camber of the upper surface, and DH to AB that of the lower surface. The actual thickness of the various parts of an aerofoil will of course be largely governed by such questions as the strength of the parts and the stresses which will be brought to bear upon them. Plate I gives a diagram showing the construction of the wing, but for the present purpose it will be assumed that such constructional considerations do not enter into the question and the effect of the three types of variations already referred to will be considered apart from this.

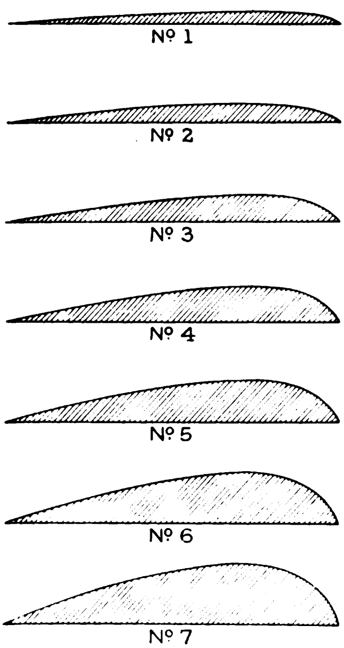
§ **3.** *Effect of change in camber of the upper surface.* One of the most important factors whose variation may effect change in the characteristics of a wing is the amount of camber which the upper surface possesses. This has clearly been brought out in investigations conducted by Eiffel and at the National Physical Laboratory. The annexed table shows that if the camber be increased continuously from zero the value of the maximum L/D reaches its highest value at a camber of 0·05 and then falls continuously. This however is only one of the characteristics that measures the effect of variation referred to above. There falls also to be considered the value of the first maximum lift coefficient and the value of L/D for low lift coefficients, chosen in the table so that $K_l = \cdot 16\ (K_l)_{\text{max.}}$, this being an average ratio used in the design of high speed machines. These are also tabulated. The sections of the aerofoils used are given in fig. 55.

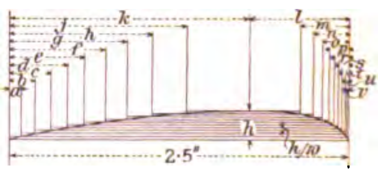
These tests were conducted at the National Physical Laboratory and described in the *Report of the Advisory Committee for Aeronautics*, 1911–12, p. 76. The size of the model was $2\frac{1}{2}'' \times 15''$ (aspect ratio 6 : 1) with rectangular plan form, and the wind speed for the test was 20 m.p.h.

Series 1.

Aerofoil No.	Camber of upper surface	Max. L/D	$(K_l)_{max.}$	L/D at $K_l = \cdot 16\ (K_l)_{max.}$
1	·0252	13·8	·48	9·5
2	·0500	14·85	·57	7·5
3	·0748	14·3	·61	5·9
4	·1000	12·7	·59	4·0
5	·1248	11·75	·57	3·2
6	·1400	11·1	·53	2·7
7	·1748	10·25	·48	1·8

It is obvious from the above results that the greatest value of the maximum L/D occurs when the camber is 0·05 and therefore, according to the requirements demanded in the preceding chapter, this one should represent the best aerofoil of the series for a climbing machine. $(K_l)_{max.}$ on the other hand attains its greatest value for the third aerofoil and should therefore be the most suitable of this set for a weight carrying machine. The last column





Inches		No. of Plane	Max. thickness h ins.
a = ·085	l = ·730		
b = ·190	m = ·350		
c = ·305	n = ·255		
d = ·430	o = ·190	1	·063
e = ·570	p = ·140	2	·125
f = ·715	r = ·100	3	·187
g = ·875	s = ·070	4	·250
h = 1·050	t = ·045	5	·312
j = 1·315	u = ·025	6	·375
k = 1·770	v = ·010	7	·437

FIG. 55

giving the criterion for a high speed aerofoil indicates that a thin one is most suitable for that purpose.

§ 4. *Effect of varying the position of maximum ordinate of the upper surface.* In addition to the considerations of the effect of varying the amount of camber on the upper surface the actual position of the maximum ordinate has been found to be of considerable importance. In order to bring this point out as clearly as possible, a series of experiments were conducted at the National Physical Laboratory and described in the *Report of the Advisory Committee for Aeronautics*, 1912–13, p. 74. The sections there used were all developed from one previously selected, by altering the position of its maximum ordinate and increasing or diminishing the other ordinates to correspond. If for example the position of the maximum ordinate was removed from the centre of the chord to a point one-quarter of the chord from the leading edge the distance between the ordinates in front of the maximum would be diminished by one-half, and those behind expanded $1\frac{1}{2}$ times. Fig. 56 gives the section where $x/c = \frac{1}{3}$. Nine such aerofoils ($2\frac{1}{2}'' \times 15''$), with flat under surfaces, were tested in the wind channel at a wind speed of 28 ft./sec. and those results bearing directly on the present question are tabulated below.

Series 2.

Aerofoil No.	x/c	Max. L/D	$(K_l)_{\max.}$	L/D at $K_l = \cdot 16\,(K_l)_{\max.}$
1	0·500	11·2	0·62	3·9
2	0·380	13	0·62	4·45
3	0·355	13·6	0·678	4·9
4	0·332	13·9	0·71	5·3
5	0·310	13·9	0·60	4·4
6	0·293	13·4	0·565	4·15
7	0·252	12·7	0·53	3·9
8	0·220	12·0	0·44	3·0
9	0·168	11·1	0·41	2·6

By comparing the figures in the three columns as before, the remarkable result appears that all three criteria of the three types of machines discussed in the previous chapter reach their best value simultaneously when the position of the maximum ordinate is $\frac{1}{3}$ of the chord from the leading edge. As a general

rule therefore any aerofoil will be improved by such a disposition of the maximum ordinate.

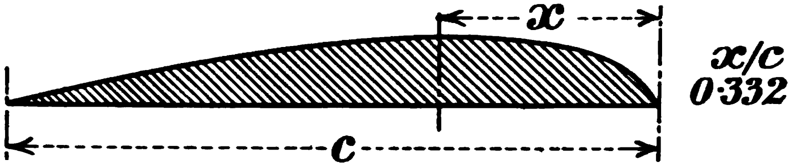
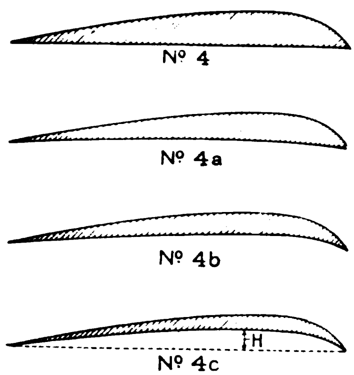


Fig. 56

§ 5. *The Lower Surface.* Very little work has actually been carried out on this question, but experimental results so far published seem to indicate that such variations do not appreciably affect the principal characteristics of a wing. This will be evident from the following table giving the results of a series of experiments (see 1911–12 Report, p. 77) conducted at the N.P.L. The aerofoil was No. 4 of Series 1, the cambers of the lower surface



No. of Plane	Top max. ordinate h ins.	Underside max. ordinate H ins.
4	·250	·000
4 *a*	·250	·050
4 *b*	·250	·100
4 *c*	·250	·150

Fig. 57

being respectively 0, 0·02, 0·04 and 0·06, and of the upper surface 0·1, see fig. 57.

It will be seen from the following table that the variations in camber have practically no effect on the values of L/D. It seems to indicate on the other hand, that the maximum lift coefficient does increase as the wing is thinned. Further experimental evidence however on this point and on the effect of the form of entry would be required to be forthcoming before any definite conclusions could be drawn.

Series 3.

Aerofoil No.	Camber of lower surface	Max. L/D	$(K_l)_{max.}$	L/D at $K_l = \cdot 16\ (K_l)_{max.}$
4	0·0	12·8	0·59	3·9
4 *a*	0·02	12·95	0·61	3·65
4 *b*	0·04	12·5	0·65	3·48
4 *c*	0·06	12·8	0·66	3·33

§ **6.** *Other Modifications in Section.* Experiments have been undertaken to determine the extent to which an aerofoil can be thickened in the neighbourhood of the rear spar, as such extra strengthening is a desirable quality from a constructional point

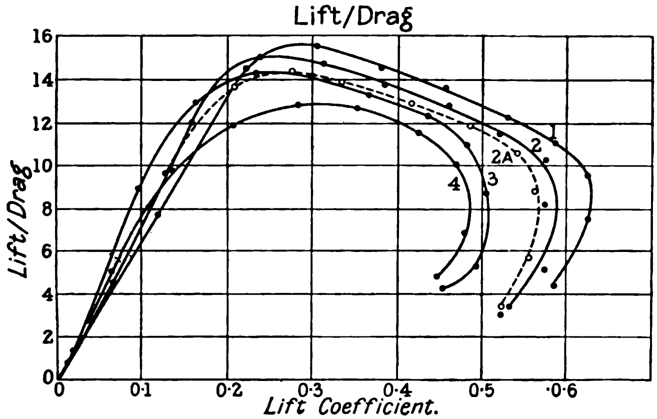


FIG. 58

of view. These indicate that very little effect is produced on the aerodynamical properties of the wing, especially if slight thickening takes place on the under surface.

Still another modification that may have a considerable effect on the aerodynamical properties of the wing is that involved in reversing the curvature of the trailing edge. Fig. 58, abstracted from Report 72, 1912–13, *Advisory Committee for Aeronautics*, gives the variation in L/D with lift coefficient for aerofoils with various degrees of reversal of curvature (fig. 59). The student will easily verify, following the lines already indicated, that such a change in curvature tends to increase the suitability of the aerofoil for high speed work. The curious effect this produces

on the variation of the position of the centre of pressure is of considerable importance, and will be discussed later in this chapter (see § 10).

It cannot be repeated too often that although attempts are always made in aerodynamic investigations to isolate the effects of any particular modification, the difficulties inherent in achieving this are always such as to throw doubt on the possibility of applying the result with rigour.

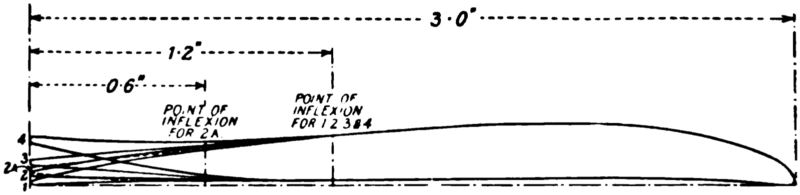


FIG. 59

2. *Variation in Aspect Ratio.*

§ **7.** The aspect ratio may be defined as the length of the span of the wing divided by the chord. That some effect should be expected by varying this quantity from wing to wing is not unnatural if it be remembered that the mere fact that the wing has ends at all will be sufficient to affect the flow, in that region at least. At some distance from the ends, the flow for all practical purposes is two-dimensional in a plane parallel to the plane of symmetry of the machine, but as the wing tips are approached the effect is to alter this state. The pressure on the under surface being moderately large, there is a tendency for the air to be drawn round the tips of wing towards the upper surface where the pressure is exceedingly low on account of the suction effect produced there by the motion. The greater the span the less will this effect be appreciated at some distance from the end, and the question at issue is whether by varying the span an increase in this effect or otherwise will operate to increase the aerodynamic efficiency of the wing.

This question has been the subject of some investigation by Eiffel and at the N.P.L., but not sufficient to lead to very definite conclusions on some points. In the *Report of the Advisory Committee for Aeronautics*, 1911–12, pp. 39 and 74, a series of experiments were described in which the same wing section,

somewhat similar to "Blériot 11 bis" with 4″ chord and rectangular form, was tested in the wind channel for values of aspect ratio ranging from 3 to 8. The results as far as they relate to the present discussion are tabulated below.

Aspect ratio	Max. L/D	$(K_l)_{max.}$	L/D at $K_l = 0{\cdot}16\ (K_l)_{max.}$
3	10·1	·670	4·75
4	11·5	·680	5·15
5	12·9	·685	5·5
6	14·0	·686	5·2
7	15·1	·689	5·0
8	15·5	·686	5·2

The value of the maximum L/D increases rapidly with increase of aspect ratio and although at 8 to 1 it is still rising, constructional reasons usually operate to limit the extent to which a designer might use this fact. The L/D at the chosen low lift coefficient is apparently not affected by change in aspect ratio to any appreciable extent, and it is not surprising therefore to find that a high speed machine rarely has an aspect ratio exceeding 4 or 5.

3. *Variation in Plan Form.*

§ **8.** Various attempts have been made to increase the aerodynamic efficiency of a wing by changing the plan form, but the advantages obtained are not such as to allow much modification for this purpose to take precedence over other considerations. The plan form may be altered for two reasons, firstly for stability purposes, and secondly to increase the aerodynamic efficiency. In certain cases these two factors may operate against each other in which instance stability takes precedence as for example in the case of the Taube, but in general stability requirements may be satisfied by other and simpler means.

Eiffel in his published researches, *La Résistance de l'Air et l'Aviation*, p. 144, describes a few investigations conducted on wings whose plan forms include those of rectangular shape with and without the corners faired and rounded off. His results indicate that a trapezium with rounded corners, the larger base at the trailing edge, is slightly more favourable than the simple rectangle.

For purposes of design where experimental results are available on aerofoils of rectangular plan form the following formulae may be used to a fair degree of approximation for obtaining the force coefficients on the same aerofoil with semi-circular tips and faired so that the sections remain similar.

Lift coefficient rounded tips
= lift coefficient for square tips,

Drag coefficient rounded tips
= Drag coefficient for square tips minus $0{\cdot}0018/R$,

where R is the aspect ratio.

§ **9.** *Pressure Distribution round an Aerofoil.* As far as the actual performance of an aerofoil is concerned, all the designer need know is the magnitude, direction and position of the resultant force for any angle of attack within the practical flying range, for on these as a basis all the remaining factors of the machine may be calculated. In general these facts may be determined with sufficient accuracy by simple wind channel experiments. There remain however many points connected with the actual design of an aerofoil efficient for special purposes which could only be settled by an accurate knowledge of the manner in which the various parts contribute by the pressures on them to the resultant forces. One of the most important of these is the extent to which the pressures upon the upper and lower surfaces affect the forces of lift and drag. Such information would immediately throw light in the direction in which research on the modifications in the wing form should be conducted.

The experiment, as usually carried out, consists of mounting an aerofoil in the usual manner in the wind channel. At various points along a section and along the span of the aerofoil very small exploring holes are made each of which is in turn connected from the inside of the aerofoil to a pressure gauge, and the pressure measured at a given wind speed and various angles of incidence. An experimental difficulty essential in the method immediately presents itself. The mere presence of a hole will be sufficient to disturb the flow in that region especially if the form of the surface has undergone any change in its immediate vicinity in boring the hole. This cannot totally be overcome but its effect may obviously be minimised by making the hole as small as possible and smoothing the surface.

The results of actual experiments of this nature are published in the *Report of the Advisory Committee for Aeronautics*, 1911–12, p. 85 and 1912–13, p. 103. The wind speed was 17 ft./sec. in the former case, and 30 ft./sec. in the latter. The method adopted for measuring the pressure was such as to furnish a direct comparison between the pressure at the point under consideration and the velocity head of the wind, and the pressures actually tabulated give the differences between the point considered and the static pressure there when the flow was undisturbed by the presence of the model.

It is evident that what is measured at each point is merely the component of the total pressure there normal to the surface, and therefore cannot include skin friction*. If by this means the pressures at every point round the aerofoil be obtained and the resultant force to which these give rise be calculated, it cannot

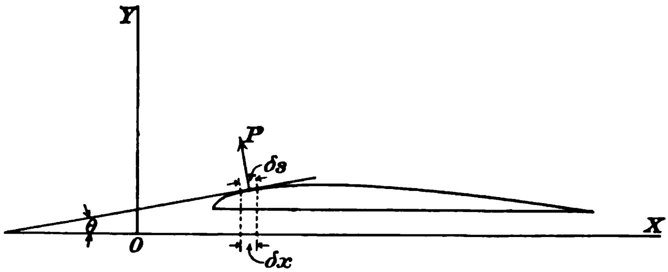


FIG. 60

contain any component of the surface frictional forces which are brought into play. The difference between the resultant so obtained and the total force on the aerofoil as usually measured in the wind channel consequently furnishes an estimate of the skin friction.

If P be the pressure as measured in the experiment normal to a small element δs of the aerofoil inclined at an angle θ to OX (fig. 60), then the amount this contributes towards the force in the direction OY is $P\delta s \cos\theta = P \times (\delta s \cos\theta) = P\delta x$ and therefore the total force in the direction OY is $\int P dx$ over the whole area. This furnishes a simple graphical means of calculating the resultant force in the direction OY to which these pressures are equivalent. It is merely necessary that the pressure measured

* See § 12, Chap. II.

at each point be plotted on an X base, and the area of the closed curve—closed since the pressures are measured right round the aerofoil—represents the component of the resultant force in the direction OY. In the same manner by plotting the pressure at each point on a Y base the component of the resultant force in the direction OX can be found. In the Report (1912–13) already referred to, OX and OY are taken parallel and normal respectively to the chord of the aerofoil. Figs. 62 and 63 represent the pressure (lbs./sq. ft. $\div \rho v^2$) as found by experiment over the central section of a wing and plotted in the manner described for angles of incidence $-2°$ to $+12°$ for every two degrees. The profile of the aerofoil used is given in fig. 61.

The areas enclosed by the curves in the first series are proportional to the force per unit length of the aerofoil at the mid-section normal to the chord, and in the second series parallel to

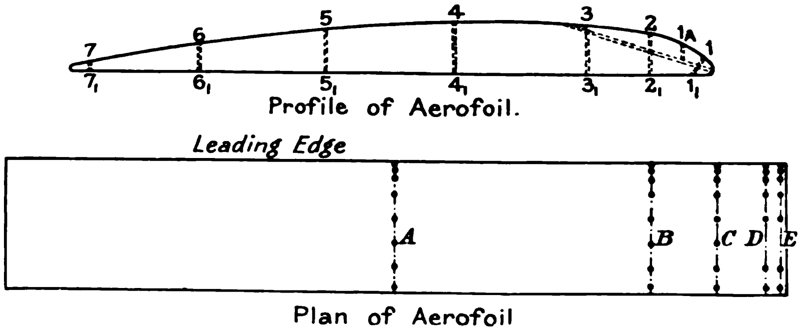
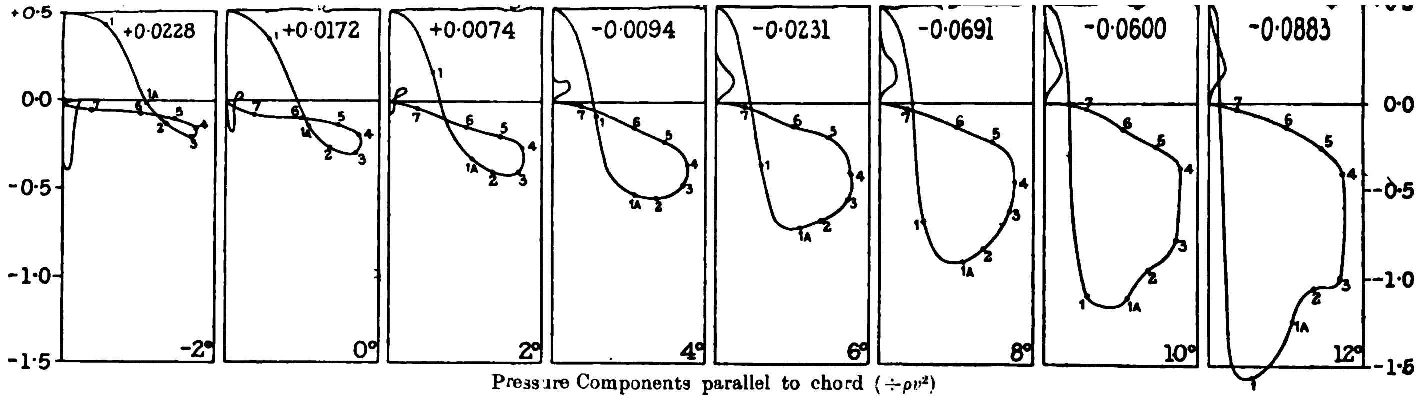


FIG. 61

the chord. It will be seen by reference to these figures how greatly the lift (practically normal force) is dependent on the suctional effect at the upper surface especially in the region of the leading edge. Between the points 3 and 4 the maximum ordinate of the aerofoil occurs and it will be at once evident how greatly the upper surface from there to the nose affects the question when it is seen that these are the points that correspond to the region of greatest negative pressure (cf. § 11, Chapter II). The minor part played by the under surface and the trailing edge is also well borne out.

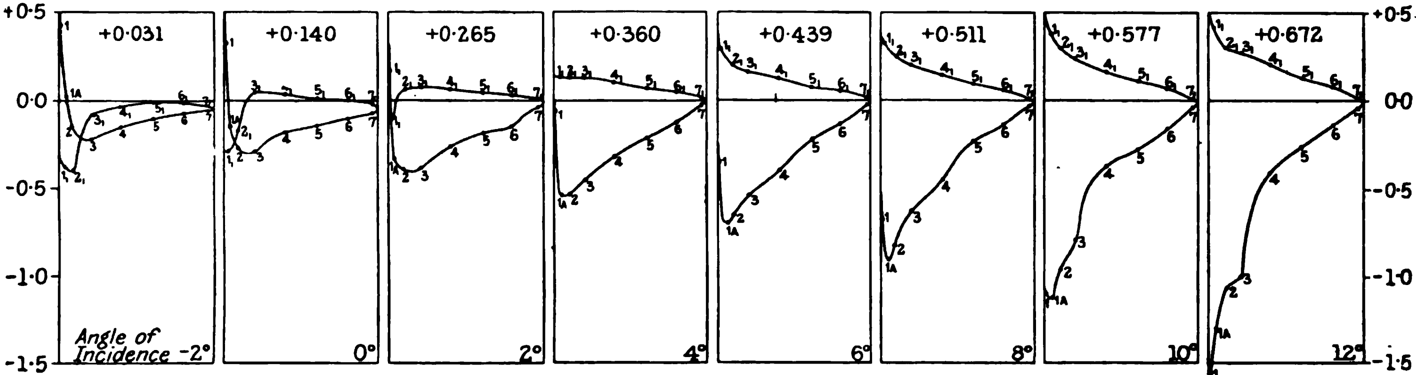
The experiments were extended to include pressure plotting for various other sections along the length of the aerofoil for the same angles of incidence. The results, integrated round each



Pressure Components parallel to chord $(\div \rho v^2)$

FIG. 62

Pressure Distribution over an Aerofoil



Pressure Components normal to chord $(\div \rho v^2)$

FIG. 63

section and expressed in absolute units, are plotted, fig. 64, at each section for lift and drag. It will be seen that the mid-section A is by far the most efficient while D close to the wing tip is the least.

The effect of the tip apparently is to reduce the lifting power in that region and the bearing of this on the question of large aspect ratio is extremely illuminating in view of the conclusions already arrived at in § 7. It may be remarked in passing, that the results in figs. 62 and 63 are of importance to the designer for purposes of strength of construction as giving the distribution of force along the wing.

Since the ordinate at any point on one of these curves represents the lift or drag, omitting skin friction, for the corresponding section and angle of incidence, the component of lift and drag due to wind pressure as distinguished from skin friction will be represented by the area of each of these curves. These divided by half the span are plotted in fig. 65, giving lift, drag and L/D for various angles of incidence. On the same diagrams the experimental values of these forces as found directly in the wind channel are plotted for comparison. The two curves for lift coincide. The drag on the other hand for angles of incidence greater than 0° is greater in the case of the ordinary force experiment, the difference between the two ordinates giving a measure of the skin friction. For negative angles of incidence the curious result appears that the skin friction is in the opposite direction. The aerodynamical possibility of this has already been referred to in Chapter II, § 12.

§ **10.** *Centre of Pressure.* The forces which are brought into play on an aerofoil will be completely determined when their positions and magnitudes are known. It is of course evident that the resultant force at any time does not act at a definite point but along a definite line, and so long as it is possible to fix some point on that line, when the components of the force are given, everything will be known regarding that force. It is customary in aeronautical experimental work upon aerofoils to select that point at which the line of the resultant force meets the chord as the point of application. This is usually defined as the *centre of pressure*, but it is in no sense a *centre* of pressure. It might in certain cases be equally, if not more, convenient to

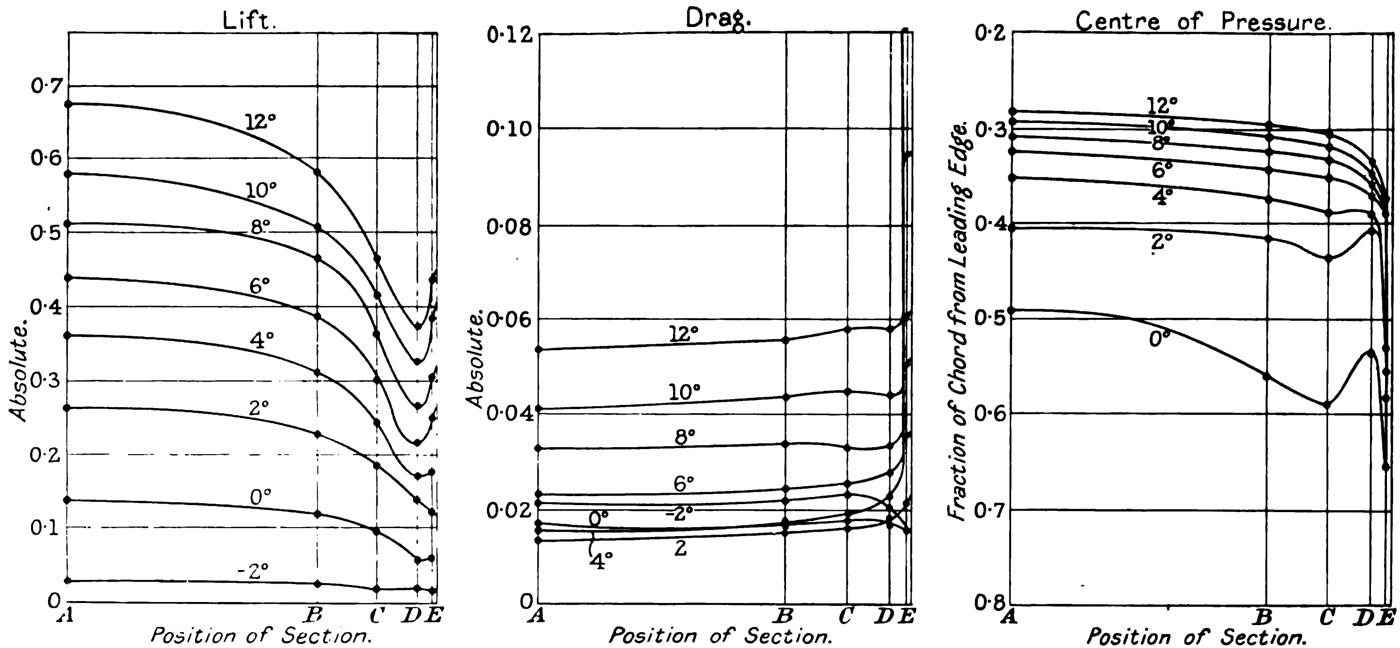

Lift.
Drag.
Centre of Pressure.
Absolute.
Fraction of Chord from Leading Edge.
Position of Section.
A
B
C
D
E
12°
10°
8°
6°
4°
2°
0°
-2°
0·7
0·6
0·5
0·4
0·3
0·2
0·1
0
0·12
0·10
0·08
0·06
0·04
0·02
0·8

FIG. 64

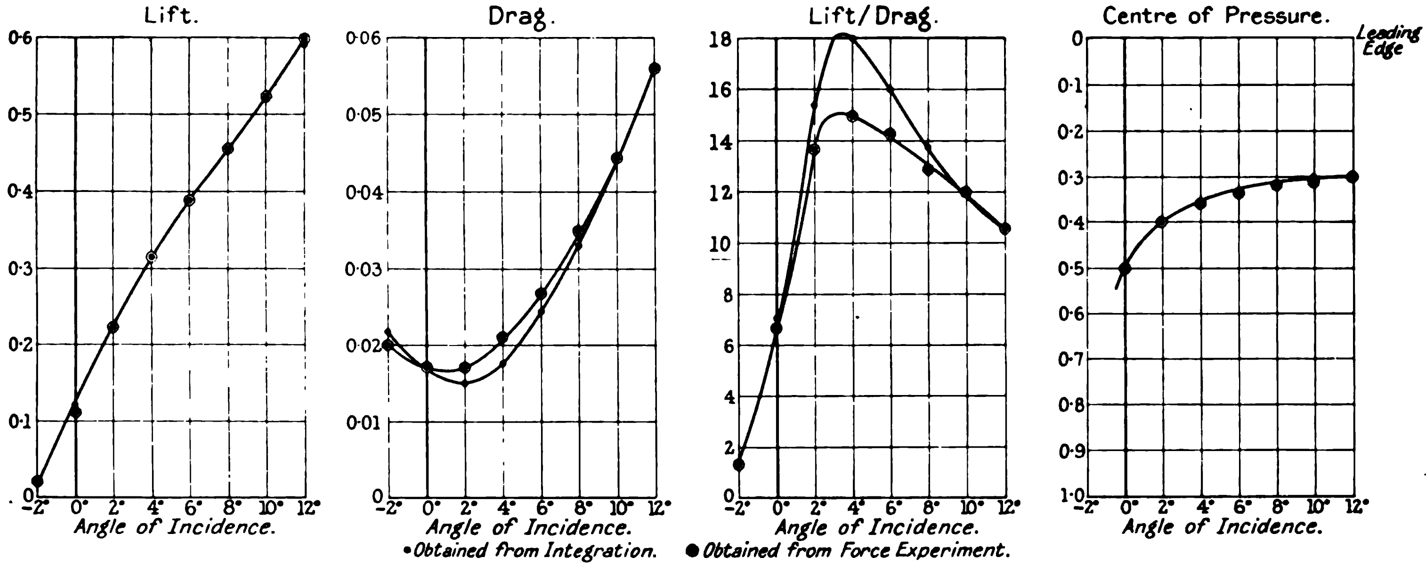

Lift.
0·6
0·5
0·4
0·3
0·2
0·1
0
−2° 0° 2° 4° 6° 8° 10° 12°
Angle of Incidence.
Drag.
0·06
0·05
0·04
0·03
0·02
0·01
0
−2° 0° 2° 4° 6° 8° 10° 12°
Angle of Incidence.
• Obtained from Integration.
Lift/Drag.
18
16
14
12
10
8
6
4
2
0
−2° 0° 2° 4° 6° 8° 10° 12°
Angle of Incidence.
● Obtained from Force Experiment.
Centre of Pressure.
Leading Edge
0
0·1
0·2
0·3
0·4
0·5
0·6
0·7
0·8
0·9
1·0
−2° 0° 2° 4° 6° 8° 10° 12°
Angle of Incidence.

Fig. 65

fix the position of the resultant force by giving its components in two directions (lift and drag) and the moment about some point, say the nose of the aerofoil, by which method indeed the centre of pressure is usually determined. It has however become the standard custom to discuss the question in terms of the centre of pressure as already defined.

Since the air forces on an aerofoil depend on the angle of attack in magnitude, position and direction, the position of the centre of pressure will vary correspondingly, and it is of extreme importance in questions of stability, and in predicting the performance of the machine generally, to know exactly how these variations take place. In particular the relation of such changes to the centre of gravity of the machine is of special significance, for upon this depends the question as to whether a change in angle of attack, when flying, will bring into operation moments about the centre of gravity tending to disturb equilibrium. A shift of the resultant force from one side to the other of the centre of gravity would change the sign of the moment, and if the centre of gravity lies in the neighbourhood of the centre of pressure a slight shift of this nature producing as it would considerable changes in the value of this moment might have important consequences. On general grounds it can be easily seen that a shift of the centre of pressure towards the nose of the aerofoil with increasing angle of attack would tend towards instability, for, in normal flight, the resultant force is practically equal to the weight of the machine, and a shift of this resultant force forward caused by an increase in angle of attack would therefore at once bring into play a couple tending still further to increase this angle. Unless such a change of attitude immediately gives rise to new restoring moments of the proper amount on other parts of the machine, for example the tail plane, a state of instability would result. It is necessary therefore to have accurate knowledge as to within what limits the centre of pressure may be expected to change its position under altered circumstances. A certain amount of such change in position may even be a desirable feature.

As already stated the position of the centre of pressure is found by determining the lift and drag, and the turning moment about some selected point. Knowing the magnitude of the two components the inclination of the resultant force to the chord of the aerofoil is at once determined. Dividing the moment by the

resultant force $\sqrt{(L^2 + D^2)}$, the leverage of the moment is found and this fixes the point where the resultant force intersects the chord, i.e. the centre of pressure. Fig. 66 gives the position of the centre of pressure for aerofoils R.A.F. 3, R.A.F. 6, and four modifications of R.A.F. 6, obtained by reversing the curvature of its trailing edge as already indicated in § 6. A glance at this figure shows at once that a reversal of curvature tends to drive the centre of pressure back towards the trailing edge with increase of angle of attack, and this as already pointed out is a desirable feature. The

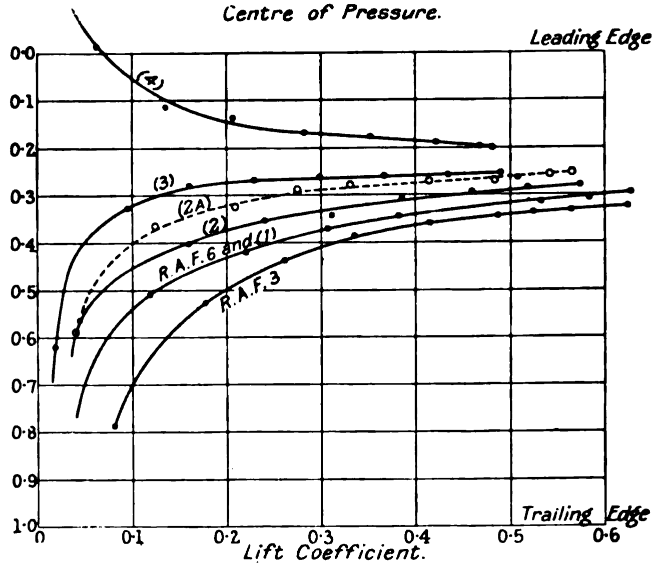


FIG. 66

authors would like once more to emphasise the further suitability of such an aerofoil for high speed machines from the point of view here under discussion.

Aerofoil Combination.

§ **11.** *Biplane.* In the attempt to increase the wing area in order to obtain the greatest lift out of an aerofoil it was found (see § 5, Chapter V) that there was a point beyond which it was not advantageous to proceed. This stage was reached when the extra weight of construction involved in an increase in wing area was just sufficient to counterbalance the increase in lift. The

method of using aerofoils in a biplane is desirable in the first place from the fact that, with a smaller loss in the necessary weight of construction, extra wing area may thus be obtained.

It is evident that the results obtained for aerofoils when tested in the wind channel singly cannot, without some modification, be applied to a biplane on account of interference effects. Such questions must be the subject of special experimental investigation, but what is the principal concern for the present discussion is how to utilise this interference in such a manner as to increase the efficiency of the combination. It is found in the first place that the wings of the upper and of the lower plane are not by any means equally efficient. The experimental method adopted to investigate these questions is to test each plane

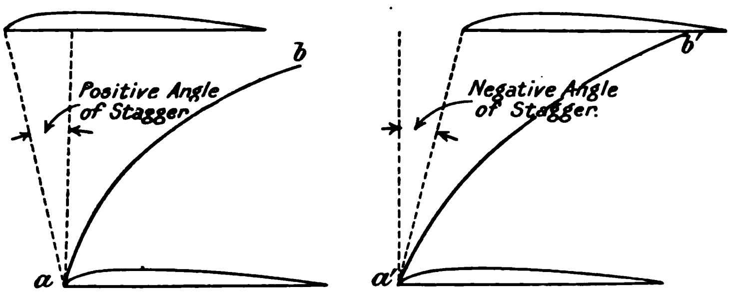


FIG. 67

separately in the wind channel with the other placed as it would be relative to it in a biplane. These indicate that the interference is such as to cause the upper plane to be more efficient than the lower, but the combination is less so than a single plane of the same total area. It is advantageous to divide the total wing area between the two planes in such a way as to increase that of the more efficient wing and accordingly the upper plane has usually a greater span. Any alteration in the relative positions of the two planes will involve a corresponding change in this interference effect.

§ **12.** *Stagger.* In fig. 67 a comparison is shown between two cases in which the upper plane in the first instance has a positive and in the second a negative angle of stagger. This angle may be defined as the inclination which a line joining two corresponding points on the aerofoils makes to the normal to the chord. It has

already been stated that in the region behind and above the aerofoil, the pressure is lower than at other points and consequently it is to be expected that when the relative position of the two planes is such as to involve a portion of the under surface of the upper plane in the region so formed by the lower plane, see b', fig. 67, the lift would be diminished. A positive stagger may be expected to yield an advantage; it remains for experiment to

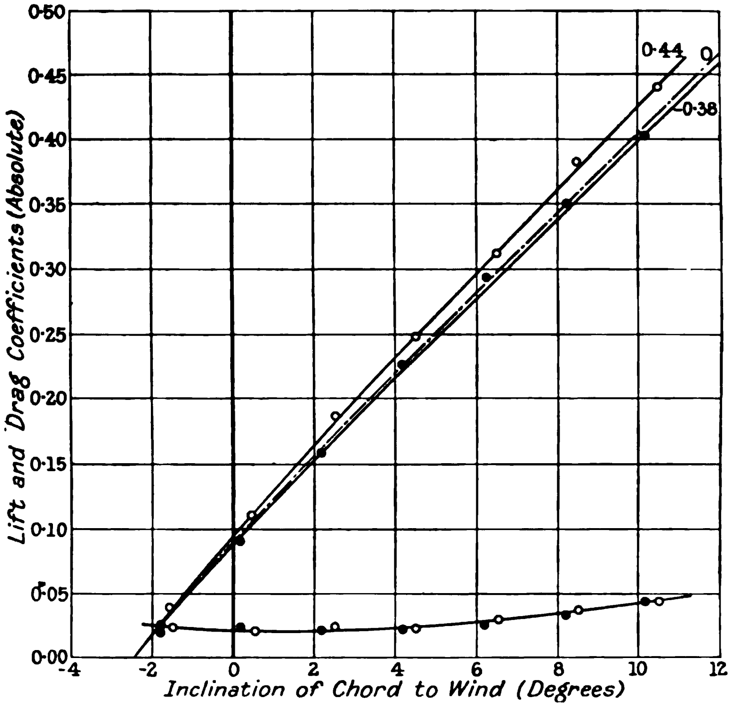


FIG. 68

determine the amount. The effect of such a modification has been investigated at the N.P.L. and published in *Report* 60, *Advisory Committee for Aeronautics*, 1911–12. In these experiments the perpendicular distance between the two planes was 5″ and equal to the chords of the planes, and tests were made when the amount of displacement of the upper plane was 0·44 and – 0·38 of the chord respectively, and the combination was tested for angles of inclination of the chords to the wind of – 2° to + 12°.

An analysis of the curves, figs. 68 and 69, shows a slight

advantage in setting the upper plane in advance of the lower. An improvement in 5 per cent. from 82 to 86 per cent. of the single plane being obtained on the L and L/D when the upper is ahead of the lower plane by 0·4 of the chord.

For many purposes, especially of a military nature, a positive stagger brings another advantage in its train. The pilot or observer usually occupies a position just above the lower plane

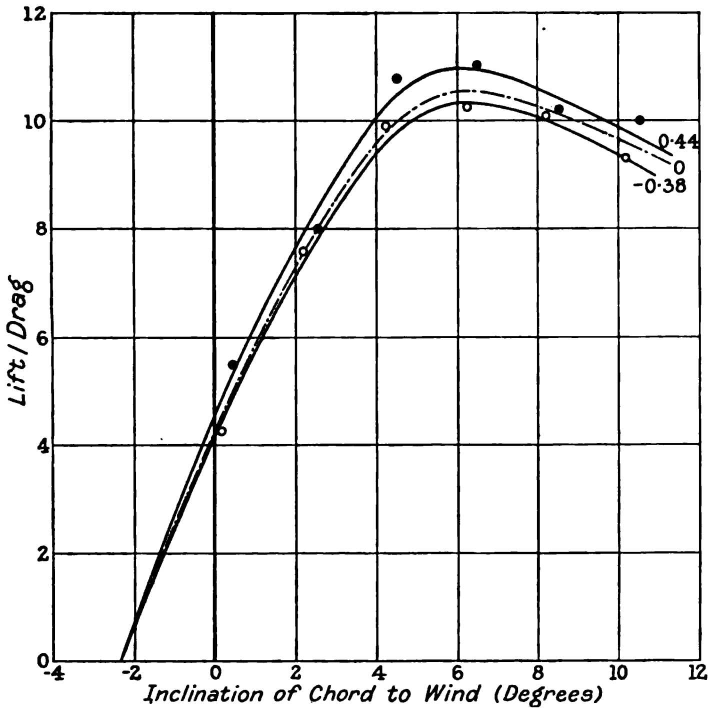


FIG. 69

and the fact that this lower wing is thrown back extends to him a wider range of vision. For warding off an attack or for observation purposes generally, this is of vital importance. Apart from the advantages enumerated above, stagger involves distinct constructional weaknesses, into which we cannot enter at present.

§ **13.** *Gap.* The interference effect between two planes may be likewise reduced by increasing the gap, usually expressed as the ratio of the distance between the planes to the chord. It is

evident that the biplane may be made to function as two single planes if the gap be sufficiently increased, but constructional and allied considerations make it more desirable to diminish the gap as far as possible. In the Report referred to in § 12 the effect of a change in gap from 0·4 to 1·6 is described. The results given in the diagrams, figs. 70 and 71, show that the lift coefficient at a given inclination is considerably less in a biplane than in a monoplane. There is a corresponding fall in drag but of smaller amount,

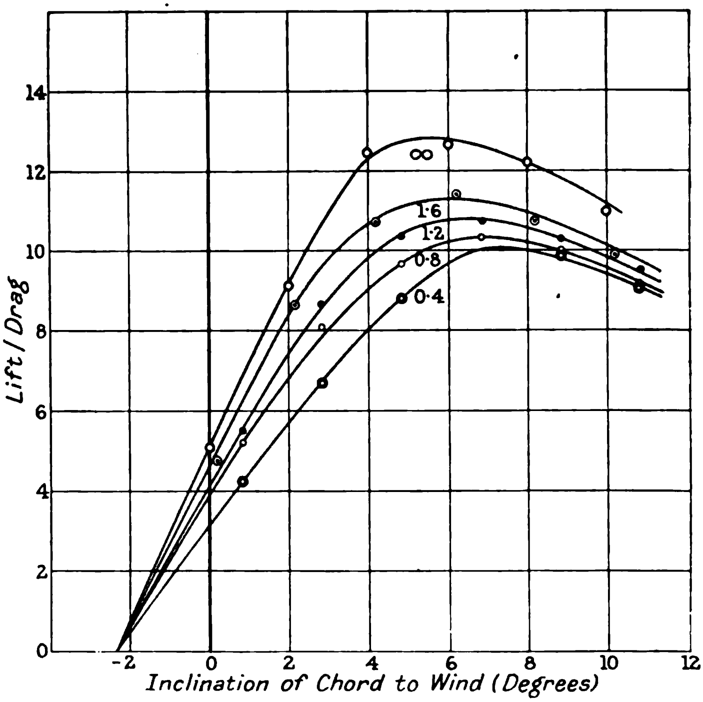


FIG. 70

so that the L/D is not so much affected as the lift itself. The curve also indicates the considerable remaining effect when the gap is equal to the chord. The normal practice is to take gap as here defined equal to 1·1.

Correction for Gap to Chord Ratio.

Let $\quad$ Gap/chord $= 1 + g$

where $\quad + \cdot 2 > g > - \cdot 2,$

then if K_l, K_d be the coefficients at gap/chord = unity
and K_l', K_d' „ „ „ „ $= 1 + g$,
then for biplane $K_l' = K_l(1 + 0{\cdot}2g)$,

$$K_d' = K_d + 0{\cdot}22g\,(K_d - 0{\cdot}014),$$

and for triplane

$$K_l' = K_l(1 + 0{\cdot}3g),$$
$$K_d' = K_d + 0{\cdot}29g\,(K_d - 0{\cdot}016).$$

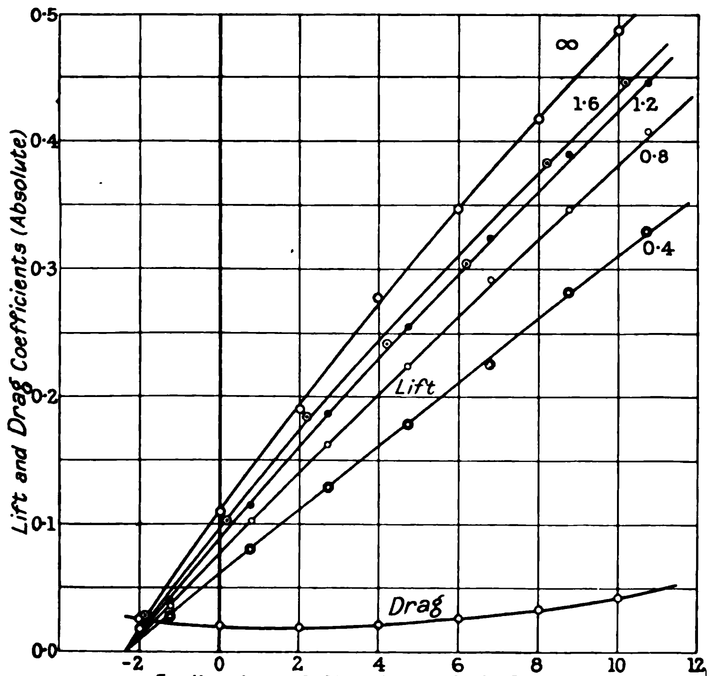


FIG. 71

§ **14.** *Triplane.* Serious attempts have been made, in the desire to obtain more wing area and therefore greater lifting capacity, to arrange the aerofoils in triplane fashion, for already, in the case of a biplane, the span sometimes extends to over 60 ft. between the wing tips, involving considerable difficulty in housing. The subject has been carefully investigated at the National Physical Laboratory and at the Aerodynamical Laboratory of the Mass. Institute of Technology, principally by

J. C. Hunsaker, and the results of the latter published in *Engineering*, July, 1916.

The triplane apparently is not so efficient as the biplane and demands considerably more engine power, but granting this it is capable of supporting nearly the same weight for the same wing area at the position of maximum lift.

Experiments conducted at the above institution to discover the distribution of load upon the three wings of a triplane (R.A.F. 6) show that the upper wing is by far the most effective of the three, and the middle wing the least. This is brought out in the following table, where the gap = 1·2 times the chord length and there was no stagger.

Angle of Incidence	Lift, upper	Lift, middle	Lift, lower	L/D, upper	L/D, middle	L/D, lower
0°	2·68	1·0	1·82	3·63	1·0	2·30
2°	2·14	1·0	1·76	3·18	1·0	2·13
4°	1·91	1·0	1·64	2·59	1·0	1·69
8°	1·56	1·0	1·36	1·49	1·0	1·37
12°	1·56	1·0	1·31	1·30	1·0	1·34
16°	1·49	1·0	1·20	1·22	1·0	1·17

Aspect ratio of each plane 6·3.
Chord 2·5″.
Wind speed 30 m.p.h.

The efficiency of the combination was tested by a comparison with monoplanes and biplanes with the same wing sections (gap of biplane 1·2 of chord), and the following table of the more characteristic points indicates the relative importance of the coefficients in these cases. The monoplane was taken as standard and the biplane and triplane coefficients expressed as a percentage of those of the former.

Angle of Incidence	Monoplane		Biplane		Triplane	
	Lift	L/D	Lift	L/D	Lift	L/D
0°	100	100	88·8	73·2	83·0	70·8
2°	100	100	83·8	74·7	75·4	69·8
4°	100	100	85·4	82·0	75·7	76·1
8°	100	100	85·2	81·9	77·4	80·4
12°	100	100	87·6	95·0	81·2	89·0
16°	100	100	98·5	124·0	96·4	145·0

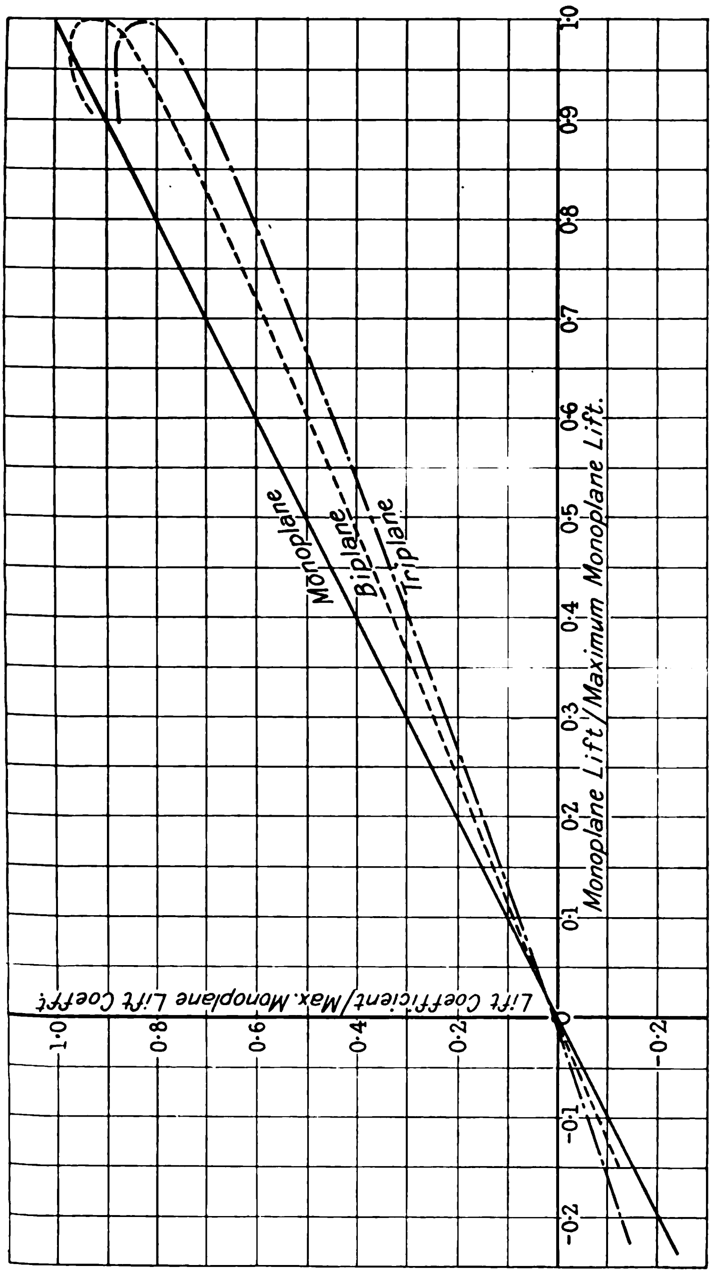


FIG 72 *a*. Lift—Biplane and Triplane corrections from Monoplane

The maximum lift of the triplane is accordingly almost as great as that of the biplane but the L/D is less. The results seem to indicate that there is a field for fruitful research in this direction and this has been borne out by the excellent performances of triplanes recently constructed.

§ 15. *Biplane and Triplane Corrections from Monoplane.* In the design of a biplane or a triplane it is always desirable if possible to obtain the results of tests on the corresponding model, but where these are not available the results plotted in figs. 72 *a* and 72 *b* may be taken as representing to considerable accuracy the corrections that must be applied to the results for the monoplane wing in order to furnish the corresponding quantities for the biplane or triplane. The figures are self-explanatory.

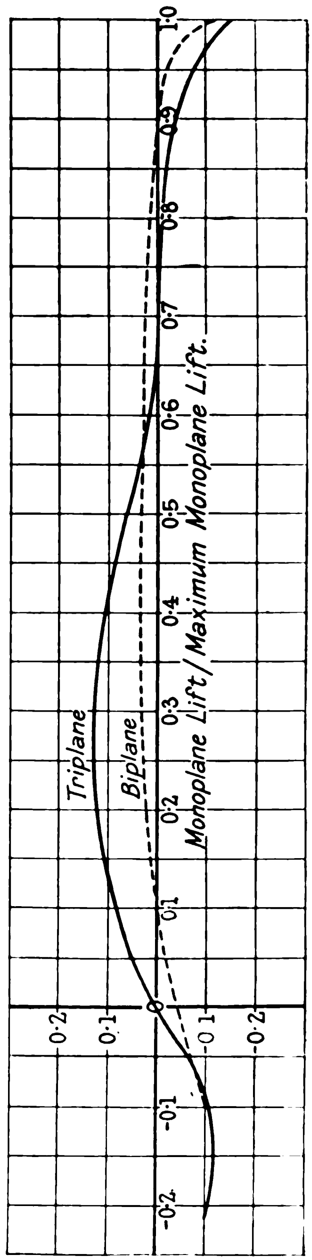


FIG. 72 *b*. Biplane and Triplane corrections from Monoplane
Vertical scale.—(Drag less Monoplane drag) ÷ Monoplane minimum drag

The position of the centre of pressure for such combinations does not appear appreciably altered from that of the monoplane if in these cases it is measured with reference to the main plane and plotted against lift coefficient.

Scale Effect.

§ 16. In a previous chapter the theoretical basis upon which depends the method of correcting the characteristics of an aerofoil for a change in scale has been developed. It was there shown

that for two bodies of different dimensions the forces brought into play upon them will be equal when the experiments are carried out at the same value of VL. Unfortunately, however, two factors at least operate to prevent a complete and direct comparison. In the first place if an experiment on a model aerofoil be carried out at the VL corresponding to the normal flight speed of a full scale wing, the forces that would require measurement—the ordinary forces in flight—would be too great for the apparatus of wind channel work. In the second place, even if this measurement were possible, the dimensions of an ordinary wind channel and the speed of wind there attainable do not give a VL of more than about one-tenth that occurring under usual flying conditions. It becomes necessary, therefore, to consider how the various characteristics of an aerofoil vary with increasing VL and if possible from these results to extrapolate to the values required under flying conditions. As a matter of fact the correction that is required in passing from experimental results obtained say in the 7′ channel at the highest VL to full scale is comparatively small, and after a little experience can be predicted with considerable precision.

Certain characteristics are scarcely affected at all by the passage beyond a certain point to larger values of VL, as for example the maximum lift coefficient and the angle at which it occurs. On the other hand the maximum L/D receives a considerable increase while the angle of no lift reaches a slightly greater negative value than for the experimental scale. It was seen previously in Chapter IV that certain values of VL represented the point of transition from one type of flow to another and were therefore critical. If it should happen that the VL for the model experiment lies in the region of this point of transition certain characteristics will occur in the model curves which by the passage to higher values of VL should be eliminated. This in point of fact does occur.

Experimentally, the method generally adopted is to carry out the investigation on the model to as large values of VL as possible by increasing the velocity and, if convenient, the size of the model until a stage has been reached where the force coefficients measured do not increase sensibly with further increases of VL. Extrapolation with small danger of error is thus greatly facilitated. In practice it is extremely convenient to plot results on

a log VL base instead of VL as this extends the region on the graph for the smaller values of VL investigated, and contracts the unknown region for larger values. A base $1/VL$ would also be convenient for this purpose.

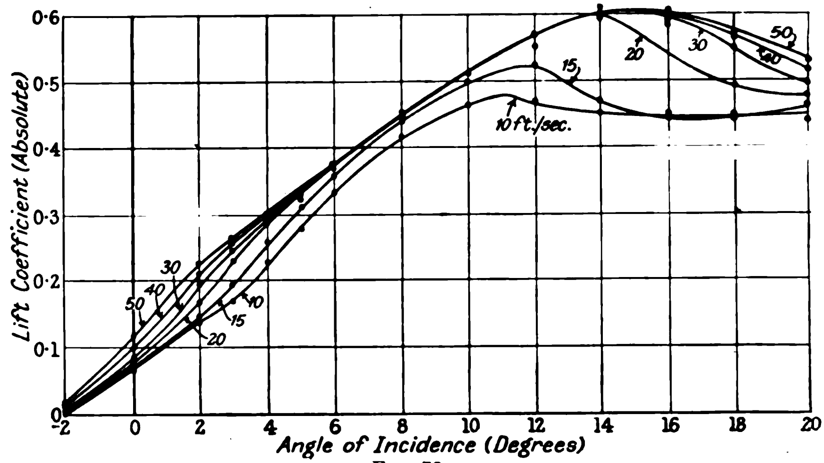


FIG. 73

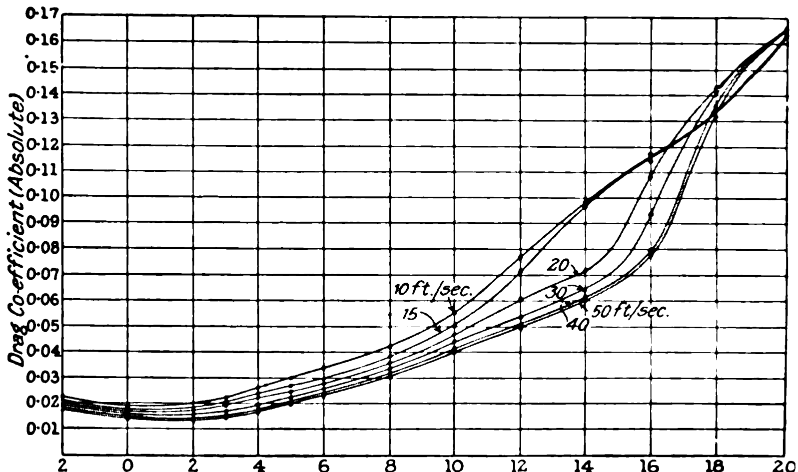
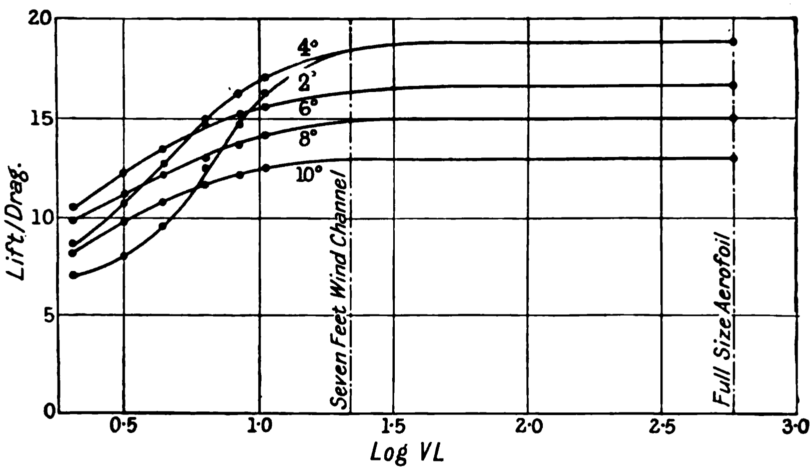


FIG. 74

A large amount of experimental work has naturally been carried through on scale effect as it is of such vital importance for calculation in design. In *Report* 72, 1912–13, *Advisory Committee for Aeronautics* (p. 78), a series of such tests on a model

of R.A.F. 6 aerofoil is described and the variation of lift and drag coefficients with increase in VL is investigated. The aerofoil was tested for inclinations of the chord to the wind varying from $-2°$ to $+20°$ and for values of VL from 2·1 to 10·4, where V is the wind speed in ft./sec. and L the chord of the aerofoil in feet. The variations of the lift and drag coefficients with angle of incidence for the different values of VL are given in figs. 73 and 74. These show the existence of distinct variations in these coefficients with VL, the most marked occurring in the value of the drag. The lift coefficients for a given inclination appear to tend to a limiting value, namely that measured at the highest speed obtainable in



Variation of Lift/Drag with VL
L = length of chord in feet
V = velocity in feet per sec.

FIG. 75

the channel. Unfortunately the range of VL is not sufficiently great for a precise prediction of the drag on a full scale aerofoil to be determined. To remedy this a comparison was made with tests on full scale aerofoils carried out at the "Institute Aérotechnique de St Cyr." Fig. 75 gives the changes in L/D plotted on a log VL base for certain angles of incidence, and there has also been incorporated in the same diagram the results obtained for the full scale experiment. The curves indicate clearly the rapidity with which the L/D obtained on the models approaches

the steady full scale value, and the error involved in extrapolating is not considerable. The curves in fig. 76, obtained from the same results, assuming a landing speed unity for the machine, show how the performance curve varies with scale. The dotted curve, estimated from the full scale results already referred to, gives the limit to which they tend. The annexed table may serve as a guide to such correction.

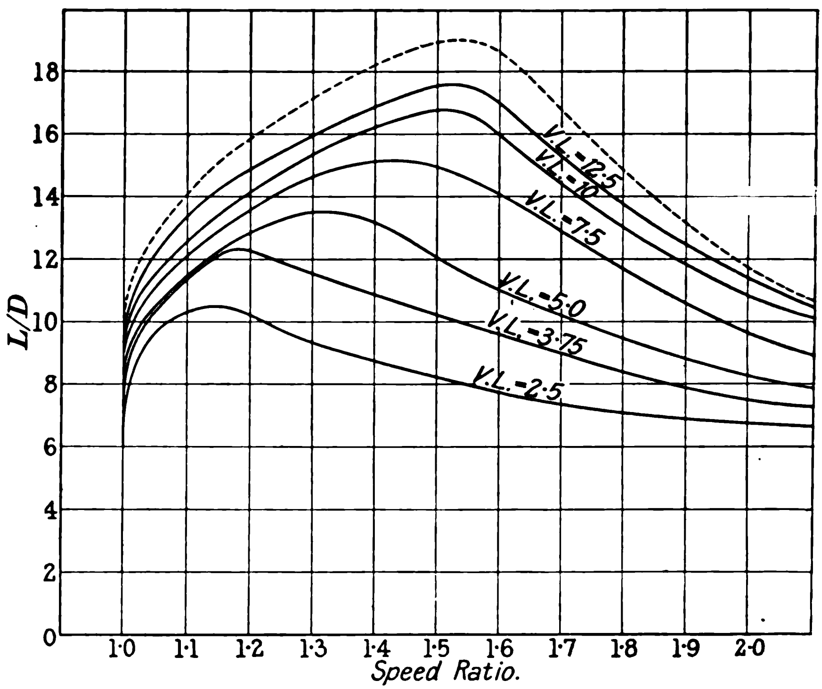


FIG. 76

The curves given here may be considered as furnishing conversion factors from which the L/D of a full scale aerofoil may be estimated from wind channel experiments on models. Accurate values for conversion could ultimately of course only be obtained from accurate experiments at the full scale value of VL, but the above discussion outlines the method to be applied and the changes that may be expected in the passage from model to full scale.

§ **17.** ***Corrections to be applied to the results of model experiments to convert into the corresponding full scale measurements.***

Aerofoil	Value of VL	Max. L/D	Angle at which max. L/D occurs	Max. lift coefficient	Angle at no lift	Angle at max. lift
R.A.F. 6	10	16·6	3·9°	0·6	– 2°	15°
Full scale	500	19·0	3·0°	0·6	– 2·8°	15°

Deductions from above table.

(*a*) Increase maximum L/D by 15 per cent.

(*b*) Diminish angle at which maximum L/D occurs by 20 per cent.

(*c*) Add to the angle of no lift – ·8°.

(*d*) Do not alter the angle of maximum lift coefficient nor the value of this lift coefficient.

CHAPTER VII

STRUCTURAL PARTS AND CONTROLS

§ 1. Every part of the machine in its motion coming into contact more or less direct with the air is acted upon by forces which will operate in some measure upon the motion. Certain of these surfaces exist and are designed purely in view of the fact that these forces are originated. This is the case for example both with the propeller and the wings. Controlling surfaces on the other hand, the tail plane and the rudder among others, owe their existence, their position and their dimensions to the fact that only under certain circumstances air forces of a specific type may be brought into being. There is another class of surface which does not fall within either of these categories. Such bodies as struts and wires whose functions are not in any sense aerodynamic, but are essentially required for the furnishing of strength to the machine regarded as a structure, nevertheless give rise to resisting forces which are not otherwise contemplated. From the point of view of the aerodynamic forces brought into play during flight, these various classes, as will be seen, practically include all the external portions of the machine, but there does not exist any clear line of demarcation between these several groups. A propeller whose sole function is to provide a forward thrust cannot do so without discharging the air behind it in a comparatively sharply defined stream. If the air propeller be fixed in front as in the case of a tractor machine, this discharged air or slip stream acting upon the parts such as the body and wings immediately in its rear disturbs the flow and modifies the forces upon them. These so-called interference effects are clearly necessarily inherent in any such combination, and must receive special investigation, and allowance made for them. It is proposed in this chapter to study the forces that are called into being by the different classes of surfaces enumerated above, excluding the aerofoil and propeller

except in so far as they give rise to interference effects, and to discover how these forces can be utilised to fulfil the aerodynamic functions, if any, for which they exist. Generally it may be stated as a fundamental principle in design that these surfaces should be so shaped as to provide the minimum sum total of resisting forces foreign to their function.

§ 2. For simplicity in study it will be convenient to consider for the present two of these groups only.

(*a*) Those parts that exist purely for constructional reasons but which also incidentally offer resistance to the forward motion.

1. The body and fuselage or outrigger.
2. The struts and wires.
3. The undercarriage.
4. The propeller when not driven by the engine.

(*b*) Those parts that exist for the purpose of providing guidance and stability to the machine—classified generally as controlling and stabilising surfaces.

Controls.

1. Rudder.
2. Elevator.
3. Wing-flaps.

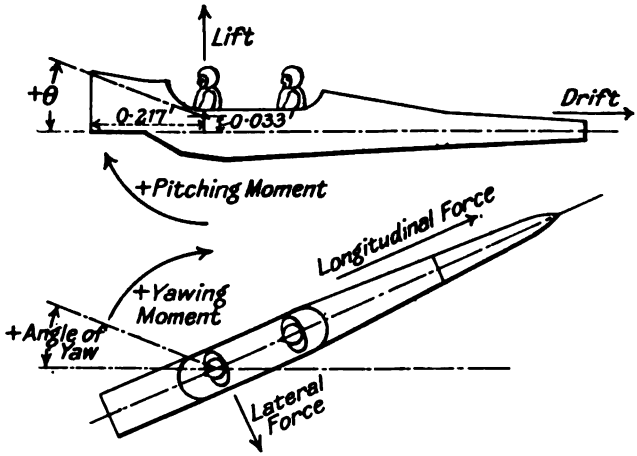
Stabilising surfaces.

4. Tail plane.
5. Fins proper and equivalent fins.

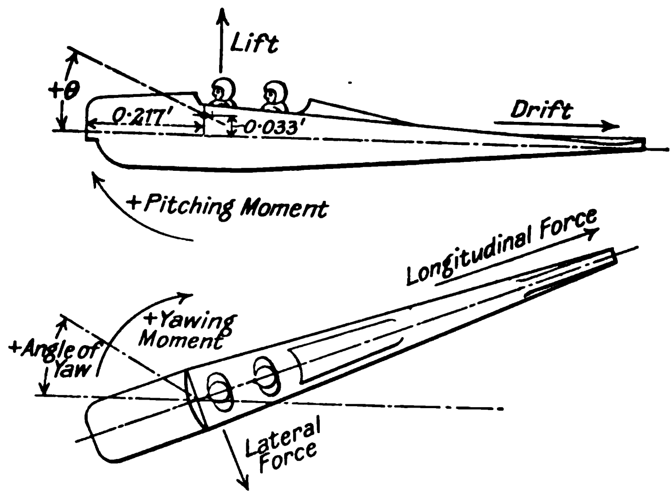
Parts existing for Constuctional Reasons but offering Resistance to Forward Motion

§ 3. *The Body and Fuselage.* The function of the body is to provide seating accommodation for the pilot and passengers, room for the instruments necessary in the determination of speed, height and climb, and a solid basis for the engine. The fuselage connecting the body and the tail system exists purely to provide support for the latter. It is an unfortunate fact that the resistance of this combination totals generally one-half that of the complete machine. In view of this the importance of the introduction of modifications in shape and the adjustment of parts likely to diminish this head-resistance is self-evident. After the discussion

of stream-lining entered into in Chapter II, it will be clear that generally the most suitable shape will be one with a comparatively



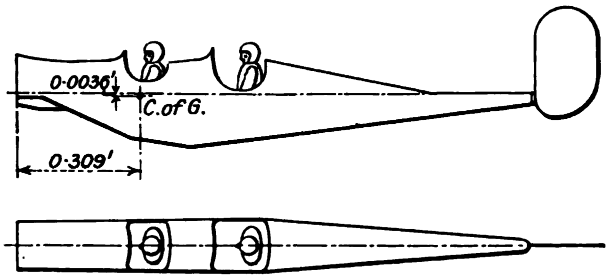
Model B.E. 2.
FIG. 77



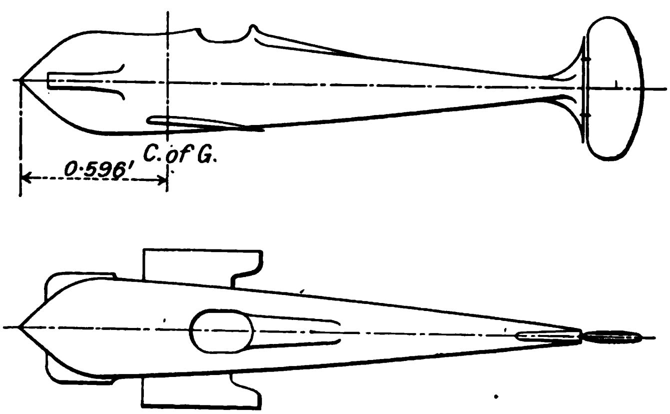
Model B.E. 3.
FIG. 78

blunted nose and a tapering body. The fuselage fortunately enters as a happy combination to supply the tapering required and as such, although it exists for a purely constructional purpose,

operates simultaneously to diminish the resistance of the whole. To determine a good shape for the body, on the other hand, serious difficulties have to be faced. In the case of a tractor, where the propeller is situated in front, the engine occupies the nose of the body, and since for the sake of cooling it must not be



Model 4.
FIG. 79



Model 5.
FIG. 80

completely enclosed, good stream-lining is made impossible. A certain amount of modification of the shape of this part is however possible, and experiments have been conducted to determine what, within the limits set by the necessary existence of various parts, is the most suitable shape. Figs. 77, 78, 79 and 80 give an outline of four bodies tested at the N.P.L., *Report* 74 *of the Advisory*

Committee for Aeronautics, 1912–13, p. 121. The results are given in the following table for the full size bodies.

Body	Drag at 60 m.p.h. in normal flight (lbs.)	Drag (length)²
B.E. 2	54·0	0·102
B.E. 3	25·8	0·041
Model 4	35·3	0·080
Model 5	18·4	0·054

The above are without rudder or elevating planes.

For bodies of the same length the above shows that the B.E. 3 body is by far the best form, its drag being less than half that of the B.E. 2.

It must be noted to the disadvantage of the tractor that the existence of the propeller in front of the body necessitates the latter lying directly in the slip stream. The velocity of outflow from the propeller is considerably greater than that of the machine relative to the air as a whole and consequently the resistance of the body is greater than would otherwise be the case. At the same time, experience has shown that the existence of the body in the slip stream tends to increase the efficiency of the propeller. It follows that these two portions of the machine ought not to be considered separately but rather their efficiency as a combination. The pusher type of machine, on the other hand, having the propeller behind, necessitates a short tail, since the fuselage no longer exists, but is replaced by an outrigger, and the resistance of the body is thereby increased. The existence of the body in front has been found, in many cases, to increase the efficiency of the air-screw slightly, but never to the same extent as in the case of the tractor. Further experimental evidence, however, on these points from the aspect of efficiency of combination of body and propeller would require to be forthcoming before the above conclusions can be regarded as thoroughly substantiated.

§ 4. *Struts and Wires.* To some extent these parts have already been discussed in Chapter II. As there explained the principle to be applied in the selection of a good strut or wire

is that the resistance should be low and its variation with angle of yaw small*. Nevertheless one distinction can be drawn between these two which modifies the selection in each case to some extent. In the case of wires the length is of course fixed from the dimensions of the machine and its cross-sectional area such as to withstand the stresses brought to bear upon them. This is equivalent to fixing the weight of the wires which must be so selected as to satisfy this requirement. With struts, on the other hand, the case is somewhat different, since struts of given length and equal strength are not always of equal weight. If W be the weight of the struts per unit length, then the extra drag of the machine due to the fact that this weight has to be supplied is $\frac{W}{L/D}$, so that if R be the resistance per unit length of the strut $\frac{W}{L/D} + R$ is the increased resistance of the machine for each unit length. For high speed machines it would therefore be advisable to select the strut consistent with the principle that the above expression be a minimum. For weight carrying and climbing machines direct economy in weight of struts is clearly an important factor.

As regards the actual shape the main general considerations have already been outlined in the chapter referred to. Of all aerodynamic bodies, struts are probably by far the most sensitive to slight changes in shape. For reliability in performance it is consequently advisable, if possible, to select one not too sensitive to such variations in view of the extreme difficulty of accurate manufacture. As a result of numerous experiments on the effect of such changes it is now recognised that too blunt a nose is a disadvantage. A semi-circular nose in fact from this point of view is about the limit of bluntness beyond which one ought not to pass. A tapering tail of almost flat contour appears to exert the most steadying influence on the flow and is consequently the best form of surface. The upper surface of the aerofoil given in fig. 49 gives the half of the symmetrical cross-section of a comparatively good stream-line strut in common use when the ordinates are increased in the ratio 1·65 : 1. Frequently for the sake of lightness struts are made with a cylindrical metal tube as

* See § 7 for definition of yaw.

a centre piece to take the load and the stream-lining effected by a fabric cover fastened in position by wooden ribs to give the requisite shape.

In practice a considerable amount of trouble arises owing to the failure of wires under vibration. Experiments which have been conducted to determine whether the resistance is increased by such motion have so far failed to detect any effect. An analysis of the forces that come into play during vibration is interesting from the fact that these are often such as to render the motion unstable and consequently maintain the vibration and even increase its amplitude. A lateral motion of this nature during flight is equivalent to an alteration in angle of incidence of the relative wind, and in such a position the lateral component or cross wind force may act in the direction of the motion of the wire. In this connection it may be remarked that in the case of most wires inclined at an angle of yaw to the wind the cross wind force is oppositely directed to what might be expected from the corresponding case of a flat plate.

§ 5. *Undercarriage and Skid.* The principles regarding stream-lining apply with equal force to all parts of the undercarriage that come into contact with the air. It is found that the resistance of the wheels is considerably decreased by covering the disc with fabric.

§ 6. *Propeller.* When not driven by the engine the propeller on account of its awkward shape offers considerable resistance to forward motion. Fig. 125, Chapter XI, shows the variation in drag on the machine with forward velocity both with and (gliding) without propeller. It will be seen in a later chapter that the effect of this increased resistance is to decrease the gliding angle. This source of trouble is of course quite unavoidable.

Parts existing for providing Guidance and Stability

§ 7. *Controlling Surfaces.* The following conventions will be adopted regarding the inclination of the machine in any flying attitude. Suppose the machine in steady horizontal flight, then a convenient set of axes of reference is OX_0 horizontal and along the direction of relative wind, O being the centre of gravity of the

aeroplane, fig. 81, OZ_0 vertically upwards and OY_0 perpendicular to these as in figure. An angular displacement about OZ_0 will give the machine an angle of yaw, about OY_0 an angle of pitch and about OX_0 an angle of roll or bank, the positive directions being as indicated, viz., X_0 to Y_0, Y_0 to Z_0, Z_0 to X_0. The corresponding moments about these axes, considered positive when such as to tend to increase the angles, are termed yawing moment, pitching moment and rolling moment respectively, cf. figs. 77 and 78.

§ 8. *Rudder.* The function of the rudder is to exert a yawing or turning moment about the vertical axis upon the machine in order to set the longitudinal axis of the latter along the direction

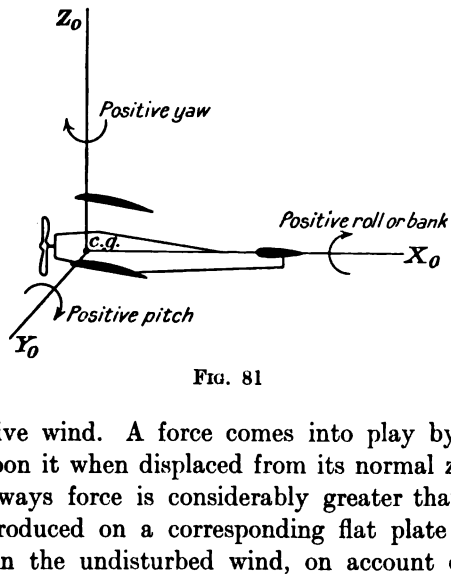


FIG. 81

of the relative wind. A force comes into play by the impact of the air upon it when displaced from its normal zero position, but the sideways force is considerably greater than that which would be produced on a corresponding flat plate of the same dimensions in the undisturbed wind, on account of the juxtaposition of the fin, the combination acquiring the nature of a cambered plane.

Balanced Rudders. With large machines the turning moment that must be obtained by the action of the rudder if of the normal type involves the production of a large moment about the axis of the rudder itself. To overcome this demands the exertion of considerable physical effort on the part of the pilot, but this difficulty has been to a great extent minimised by the principle

of balancing. The method is to arrange a portion of the rudder A, as in fig. 82, to be just above the fin, the axis being between this portion and the rudder proper. By this means the moment about the axis of the rudder, the strain of which has normally to be borne by the pilot, is reduced by the opposing moment due to A.

Limitations in Balancing. The extent to which balancing may be utilised to assist the pilot to obtain the necessary deflection without over-exertion is from one point of view subject to a serious restriction. If the moment about the hinge be plotted against the angle of deflection of the rudder, fig. 83, a curve of the type OB is normally obtained, O being the origin, so that if

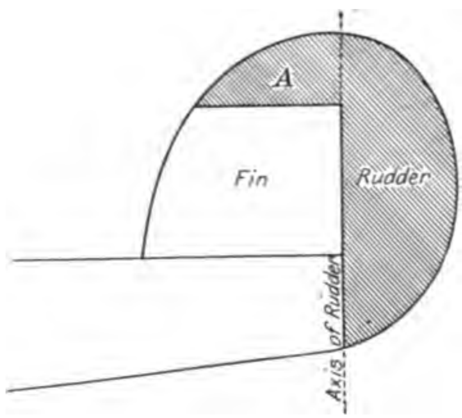


FIG. 82

OA represents the moment the pilot is capable of exerting, OL will represent the deflection he can secure. By balancing in the manner indicated the moment to obtain a given deflection is reduced and a curve of the type OC obtained, where OM now represents the deflection for the moment OA. At O the rudder is in equilibrium, the moment being zero, and along OC the moment is such as to oppose any deflection. Under these circumstances the zero position is one of equilibrium and of stability. The limiting curve of the class of this type occurs when OC just touches the horizontal axis at O. This will provide the maximum deflection OM that the pilot by his own exertions aided by balancing is able to produce, for, the next curve providing a greater deflection will

be of the type $E'F'OFED$. Here O is a position of instability, since when the rudder is in normal position the moments caused by a slight deflection are such as to increase the latter, which is evidently an undesirable feature.

§ 9. *Elevators.* The elevator is capable of rotation about an axis perpendicular to the symmetrical plane of the machine, and as such can be utilised for effecting an alteration in the value of the pitching moment on the aeroplane. There are two functions which the elevator may be called upon to fulfil in virtue of this fact. It may be used in the first place for correcting any small variation in angle of pitch by introducing by the appropriate deflection a pitching moment in the requisite direction. It is, in

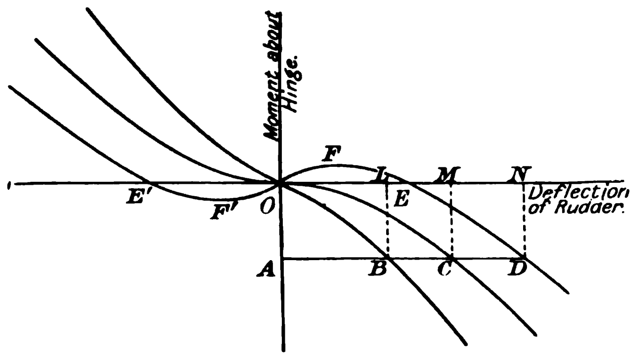


FIG. 83

the second place, the most effective means of regulating the speed of flight, for an adjustment of the elevator will cause an alteration of the pitching moment on the machine and accordingly its attitude will alter until a position of equilibrium is reached where the total pitching moment is zero. This will of course occur at a special value of the forward speed. The setting of the elevator thus determines the speed of flight. As in the case of the combination of fin and rudder so the tail plane and elevator pertain to the nature of a cambered plane and are thus more effective than they would be singly in producing the requisite moment.

At high angles of attack of the machine the elevator is of course in general inclined upwards and at very low angles of attack inclined downwards. In both these extreme positions a

considerable effort requires to be exerted continuously by the pilot to maintain the control in its appropriate position. At the same time it is necessary to allow him a certain amount of further play with the elevator for correcting in each of these positions small variations in pitch. Two methods have been adopted in face of this difficulty. In the first place the elevator has been balanced in a similar manner to that effected with the rudder, but as before limitations are imposed on the extent to which this may be utilised. In addition it must be remembered that although the balancing may be effective at one angle of attack it will not necessarily be so for all. This difficulty does not arise to the same extent in the case of the rudder. In the second place the function of the elevator to fix the speed of flight has been transferred to the tail plane by making the latter adjustable, and the elevator proper has been reserved purely for the purpose of eliminating small fluctuations in pitch.

§ **10.** *Wing Flaps.* The purpose of the wing flaps is to introduce a moment tending to turn the machine about the longitudinal axis OX, thus originating the angle of bank required for circular flight or for correcting an accidental roll. A setting of this control corresponds to a downward deflection of the wing flaps on one side of the machine and an upward deflection of those on the other, producing an increased lift on the one wing and a decreased lift on the other. The same considerations adduced in the case of rudder and elevator with regard to balancing apply equally here.

Since the function of the aileron or wing flap is to produce a rolling moment about the longitudinal axis of the machine, the efficiency of this part for a particular angle setting may be regarded as the ratio of the rolling moment produced to the moment about the hinge of the aileron. The latter is the moment that must be exerted by the pilot. Using this definition of efficiency, it is found experimentally that the ailerons are most efficient near their undeflected position falling off gradually with increasing angle until at about 15° the efficiency suddenly diminishes owing to a sudden drop in the rolling moment. For this reason it is never necessary to exceed this angle. In order to obtain high efficiency, ailerons of large span should be used. For a depth of aileron ·22 chord and length twice the chord the following

figures may be taken as a guide within the range of angle of aileron + 12° to − 12°.

Angle of Incidence (degrees)	Rolling Moment Coefft. per degree aileron deflection	Hinge Moment Coefft. per degree deflection
0°	0·0164	0·00104
8°	0·0172	0·00084
16°	0·0126	0·00084

where rolling moment coefficient $= L/\rho ASv^2$,
hinge moment coefficient $= H/\rho ACv^2$,

where $S =$ span of biplane,
$A =$ total area of ailerons,
$C =$ chord,
$L =$ rolling moment,
$H =$ hinge moment.

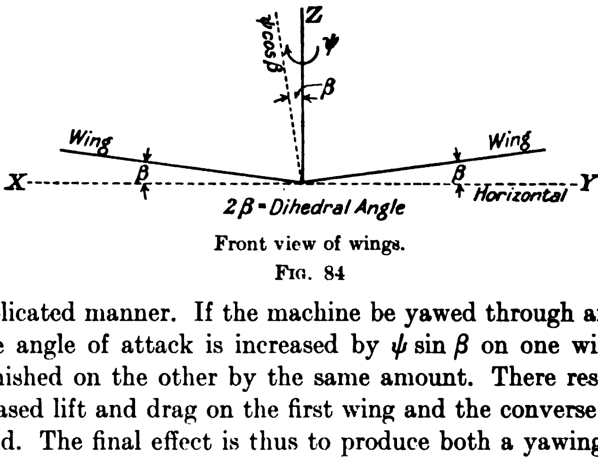
Stabilising Surfaces

§ **11.** *Tail Plane.* When the aeroplane has communicated to it a positive angle of pitch, the tail plane lowers and meets the oncoming air and by the interaction brings into being a negative restoring moment tending to right the machine. For a more accurate and detailed discussion of the operation of this stabilising surface the student is referred to Chapter XII. The action of the tail is materially affected by the downward current of air deflected from the main planes. In effect this deflected current is equivalent to an altered angle of attack of the tail plane. Experimentally it is found that the so-called angle of downwash or inclination of the relative wind at the tail to the direction of the wind relative to the machine is equal to half the angle of attack of the main planes measured from the position of no lift.

§ **12.** *Fins and Equivalent Fins.* Just as the tail plane existed for the purpose of introducing a restoring moment when the machine was pitched so do the tail fins exist to oppose an angular disturbance about the axis OZ by the introduction of a contrary yawing moment. Other parts besides the fin itself act upon the machine in a similar manner and an accurate estimation of the effects of these must be obtained before the dimensions of

the fin proper can be determined. It will be seen in a later chapter that too large a fin area unfortunately operates to produce a particular type of instability. Any part of the machine which has a cross-section of moderate size with regard to the relative wind originated during the production of an angle of yaw will act as a fin by introducing a restoring or disturbing moment roughly proportional to the product of its equivalent cross-section and its distance from the axis of yaw. The main parts of the machine which function in this manner apart from the fin itself are (1) body and undercarriage, (2) dihedral angle of wings, (3) propeller. In the undercarriage it is chiefly in the wheels covered in by canvas that fin effect is experienced.

By the dihedral angle ($= 2\beta$) is usually understood the difference between the angle contained by the wings and two right angles. As a fin the dihedral angle operates in a comparatively



Front view of wings.

FIG. 84

complicated manner. If the machine be yawed through an angle ψ the angle of attack is increased by $\psi \sin \beta$ on one wing and diminished on the other by the same amount. There results an increased lift and drag on the first wing and the converse on the second. The final effect is thus to produce both a yawing and a rolling moment so that the equivalent fin may be regarded as situated in general in the plane of symmetry above the machine and forward of the centre of gravity. In addition to such effects being produced by mere angular displacements, rotational velocities will operate in a similar manner. The question will be further discussed in the chapter on stability.

Fin effect of Propeller. Suppose the propeller blade in its rotation to be influenced by a sideways wind of velocity u, then a portion of the propeller which was previously moving with velocity v has in one position a relative speed $v - u$ and in the

other $v + u$. There will consequently arise lateral forces on the two blades not balancing each other as previously but with a resultant lateral effect. The question is not quite so simple as this, since there is a change in angle of attack at each portion of the blade with a consequent change in forces that arise. Generally this lateral force produces two distinct effects. As a fin it operates to produce a yawing moment by the direct action of the lateral force referred to, and in addition, owing to the difference in thrust on the two blades, it will produce a pitching moment.

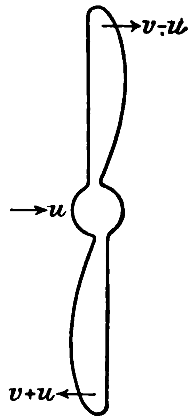


FIG. 85

At an angle of yaw of 5° the lateral force on the propeller of the B.E. 2 C is about 8 lbs. and a corresponding yawing moment of 50 lbs.-ft. The total lateral force and directional moment at the same angle, exclusive of the effect caused by the rudder and propeller, are approximately 53 lbs. and 81 lbs., so that the propeller effect is not by any means negligible. At the same time the pitching moment produced is about 123 lbs.-ft., enough to pitch the aeroplane through on angle of ½° assuming the elevator fixed.

PART III

CHAPTER VIII

STRENGTH OF CONSTRUCTION

ANALYSIS OF STRESS DISTRIBUTION IN AEROPLA[illegible] MEMBERS

§ **1.** The requirements that an aeroplane has to fulfil, e[illegible] in the case of large machines, and those designed for rap[illegible] demand as great economy as possible in dead weight [illegible] struction, and only an accurate analysis of the forces bro[illegible] play, and their effects, can indicate where this saving [illegible] effected without material weakening of the strength [illegible] to withstand the forces of flight. In general for purpose[illegible] it is not the normal forces of flight that must be consi[illegible] the maximum forces that the aeroplane may be calle[illegible] withstand under exceptional circumstances. The great[illegible] the less and this ensures that the aeroplane will c[illegible] through all its manoeuvres.

The stresses that an aeroplane must undergo are [illegible] causes, (*a*) inertia forces, (*b*) wind forces. In the fir[illegible] included all those which come into existence under t[illegible] tion of the machine or of its parts (engine and prop[illegible] example when starting, [illegible]ing and turnin[illegible] (*b*) [illegible] hand includes such for[illegible] and drag [illegible] wi[illegible] on the tail. We shal[illegible] momen[illegible] entirely to the forces [illegible]lass (*b*[illegible]

§ **2.** The maximum [illegible] into play, and those wh[illegible] as guides in design, co[illegible] flight changes rapidly. [illegible] flattening out of the fli[illegible] increased angle of attac[illegible]

increase the load on the tail even to a much greater extent. These forces will call into existence in the various members of the structure (spars, struts and wires) tensions and compressions which each corresponding part must be specially designed to withstand. The load on the tail in the same way is distributed through the longerons, struts and wires in the fuselage, in the case of a tractor, while in the case of a pusher type in the outrigger. For the present attention will only be directed to an estimation of the distribution of the forces and their effects among the members composing the wing structure and tail system of a biplane.

Nature of external loading due to wind Forces. Forces on Wings.

§ **3.** To determine how the external wind pressures affect the stresses in such an aeroplane member as a wing, it is not sufficient to know that the load on that member is simply the resultant wind pressure, but its law of distribution both cross-wise in the direction of the span, and length-wise in the direction of the chord must also be known with fair accuracy. For on the nature of this distribution will depend whether the front or rear portions are to bear the major part of the load, and which sections of these front and rear portions will be more heavily stressed.

Reference has already been made, § 9, Ch. VI, to experiments investigating the distribution in wind pressure over a wing surface. These experiments have been carried out on models and as already indicated in the earlier parts of this book must be corrected for scale effect when applied to the full-size wing. As far as the present discussion is concerned, however, two considerations relative to this are worthy of note. In the first place one does not require to derive the actual values of the wind pressures on a full scale wing from these model tests but merely their relative values, for from a knowledge of this and the total load on the wing derived from considerations of weight the pressure along the span may be determined. In the second place it is probable that at higher values of vl than those of the experiment the relative distribution may alter slightly, but for purposes of stress calculation inaccuracies that might be introduced from this cause are of comparatively minor importance, and are fully covered in this case by the safety factor. A considerable departure from the assumed law of distribution would require to take place before its effect would become apparent in the stress distribution.

§ 4. Looking at the longitudinal distribution of pressure (fig. 64, Chapter VI), it is evident that for a wide range in the vicinity of normal angles of flight the pressure is fairly uniform from the centre outwards, but in the region of the tip it falls off considerably. In determining the stresses in the framework, therefore, it will be a sufficiently close approximation to the truth to assume that the distribution in load along the span is constant, up to a short distance from the tip. From there onwards it is again supposed constant, but of smaller amount than on the remainder of the wing. Usually the total normal load is obtained from a knowledge of the landing speed as explained in Chapter V, and the distribution by the above method.

These wind pressures acting directly on the fabric of the wing are transmitted through the ribs to the two main members of the wing, the spars, but the extent to which this external loading is distributed between these two members depends entirely upon the circumstances of flight. For purposes of design where abnormal conditions must always be considered, there are three principal manoeuvres which fall to be considered:

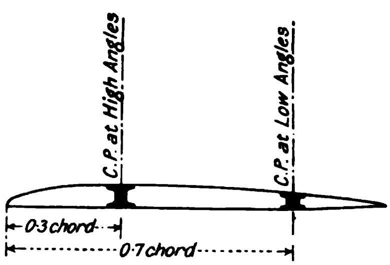


FIG. 86

(*a*) High speed flight where the angle of attack is usually very small.

(*b*) Diving vertically where the normal force on the wings is practically zero since there is no component of the weight in that direction. A vertical dive takes place therefore approximately at the angle of attack for no lift.

(*c*) Flattening out immediately after a steep dive when the angle of attack is consequently high.

It is the angle of incidence at which each of these manoeuvres is executed which determines the distribution of pressure between the two spars. A consideration of curves such as fig. 66 indicates that practically without serious error, for all sections the lines of centre of pressure may be taken as parallel to the spars and that for case (*a*) the centre of pressure to all intents and purposes coincides with the rear spar, while in case (*c*) with the front spar,

fig. 86. In both these cases the direction of the loading is principally perpendicular to the chord, the longitudinal force being of practically little consequence. The law of variation of normal force along the wing is practically the same as that of the lift given in fig. 64. In case (*b*), however, where the lift force is practically non-existent the loading is almost entirely parallel to the chord, and may be assumed for practical purposes constant along the span.

In the case of a biplane account must be taken of the fact that the upper wing contributes more to the lift than the lower plane. While practically the same law of distribution in pressure may be taken in both cases, the component amounts of lift may be taken from the following table, where the spans of the two wings are equal.

Ratio of lift on Plane to total lift of combination.

Angle	+ 30° Stagger		Zero Stagger		- 30° Stagger	
	Upper Plane	Lower Plane	Upper Plane	Lower Plane	Upper Plane	Lower Plane
- 6°	0·537	0·463	0·488	0·512	0·429	0·571
- 4°	0·504	0·496	0·503	0·497	0·452	0·548
- 3°	0·458	0·542	0·513	0·487	0·488	0·512
- 2°	0·249	0·751	0·572	0·428	0·476	0·524
- 1°	0·660	0·340	0·428	0·572	0·000	1·000
0	0·630	0·370	0·510	0·490	0·391	0·609
1°	0·608	0·392	0·482	0·518	0·392	0·608
2°	0·616	0·384	0·514	0·486	0·409	0·591
3°	0·588	0·412	0·505	0·495	0·427	0·573
4°	0·572	0·428	0·510	0·490	0·439	0·541
6°	0·573	0·427	0·513	0·487	0·452	0·548
8°	0·573	0·427	0·518	0·482	0·457	0·543
10°	0·576	0·424	0·523	0·477	0·459	0·541
12°	0·568	0·432	0·532	0·468	0·497	0·503
14°	0·511	0·489	0·532	0·468	0·564	0·436
16°	—	—	0·498	0·502	0·492	0·508

Analysis of Wing Structure for Stress Distribution.

§ **5.** The previous discussion has indicated the nature of the external wind loading which the structure has to support. Before proceeding to an analysis of the distribution in stress set up in each of the constituent members it is clearly necessary to lay down the principal geometrical features of the structure in as far as they are vital at all in distributing the stress. In a biplane the four

principal members are the spars two in each wing, connected horizontally by ribs a certain definite specified distance apart, and vertically by gap struts more or less rigidly fixed to the spars. In addition to these there exist bracing wires of various types (see Plate I). In essence this is the rigid structure loaded in a manner already described which has to be analysed for stress distribution. In the main each member will be subjected at each point to a thrust or tension and to a bending moment. It is our object to indicate how these may be evaluated.

At the outset, certain fundamental simplifications may be immediately introduced. In cases (*a*) and (*c*) it has been seen that the external loading was practically normal to the wing and situated on one or other of the spars. In that case the ribs may be omitted as far as their direct effect is concerned and the structure

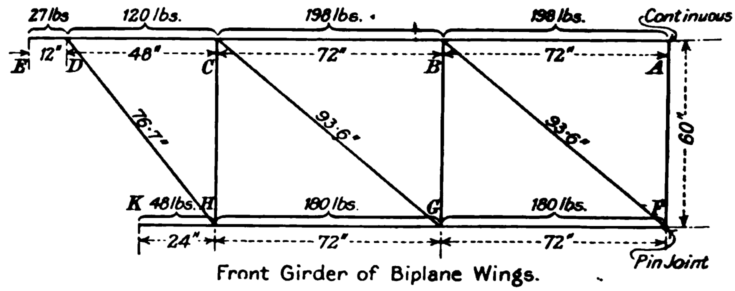


FIG. 87

may be treated as consisting either of the vertical frameworks of which the two front spars and their gap struts and wires are the principal members or of the two rear spars etc., as the case may be. For the present the incidence wires, *i.e.*, the wires connecting diagonally the front spar of one wing with the rear spar of the other, will be neglected. But the manner of allowing for their effect will be dealt with later. The problem thus reduces itself to the determination of the stress distribution in a plane framework loaded in a specified manner.

§ **6.** Fig. 87 represents such a framework where the spars are parallel and no dihedral angle exists. For simplicity the width of the body will be neglected and the complete structure will be supposed symmetrical about AF, the upper spar being continuous at A and the lower spar pin-jointed at F. The loading derived from

a consideration of the pressure distribution over the wing and a knowledge of the weight of the machine will be supposed uniform along each bay *AB*, *BC* etc. and of the magnitude given in the diagram.

The continuity of the spars at *A*, *B* etc. render the direct evaluation of the stresses a difficult operation but by a method of successive approximation as accurate a result as is required can easily be obtained. It has been found in practice that the direct forces, tensions and compressions, in the various members of the framework can be found to a fair approximation by supposing the joints *A*, *B* etc. to be pin-joints and the spars not continuous there. The lateral loading on the spars can then be replaced by concentrated loads at the pin-joints equally divided between the two supports. The overhangs *DE* and *HK* must be placed by a reaction

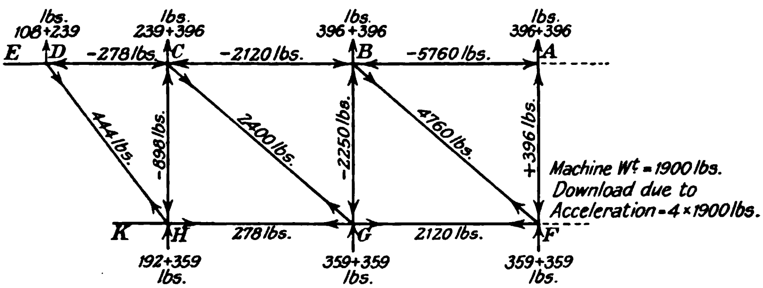


Fig. 88

at the joint *D* or *H* equal to the total load on the portion. The direct forces in the members of the structure can then be found by the polygon of forces for each joint or some equivalent method such as by Bow's notation. Fig. 88 gives equivalent external loading derived as stated above for the case where the machine is recovering from a steep dive, the loading being four times normal together with the forces in the members of the frame in fig. 87. The negative sign indicates compression.

As an exercise the student should evaluate the case where the dihedral angle is $2\frac{1}{2}^\circ$ the value usually adopted in practice. In that case the spars are inclined upwards towards the tip $2\frac{1}{2}^\circ$ to the horizontal, the gap struts as before remaining vertical and 60 inches long, *AB* etc. 72 inches long while the wires are altered accordingly.

The reaction and forces in the members determined as above must of course be modified on account of the continuity of the spars at the joints *A*, *B*, *C*, etc. Before this effect can be found it will be necessary to determine the deflections of the structure at these points because, as can easily be seen, these deflections will affect not merely the force distribution in the frame to a slight extent but also the bending moment diagram for the spars.

§ 7. *Determination of Deflected Position of Structure.* The deflected position may easily be obtained graphically by the following method. Let *ABCD*, etc., fig. 89, be the original structure, in which each of the members, ties, etc., are represented in their corresponding tensions or compressions, the values of which have already been estimated approximately.

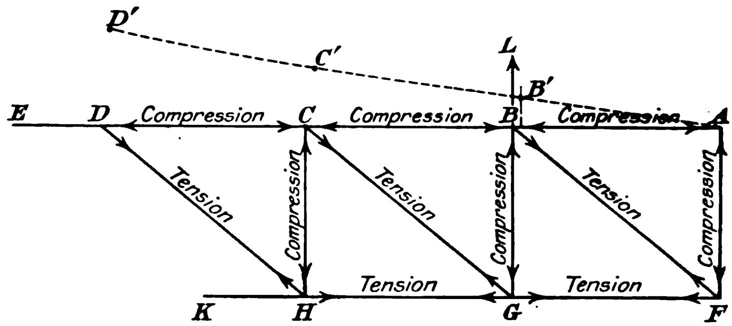


FIG. 89

From a knowledge of the cross-sections of the materials of each of the members, the actual compressions and extensions are obtained. It will be supposed that *A* the central position of the upper wing is fixed, and that *F* will be deflected to some point along the original line *AF*. Starting at *a*, fig. 90, let *af* represent in magnitude and direction the deflection of *F* relative to *A*. To find the deflection of *B* relative to *A* let *f*2 be the deflection of *B* along *FB* and *a*1 that of *B* along *BA*. Drawing 2*b* and 1*b* perpendicularly to *f*2 and *a*1 respectively, *ab* will represent the total deflection of *b* relative to *a*. Strictly 2*b* ought to be a small arc of a circle of radius *FB*, but to the approximation here required this is represented as a straight line.

For similar reasons 1*b* is also rigorously not a straight line. The method of compounding the component deflections can thus

easily be followed through from the diagram and the vertical deflections of B, C and D from A, and G and H from F, are obtained as indicated. It should be noted that fig. 90 is not drawn to scale. $B'C'D'$, fig. 90, represents the deflected position of BCD, and from the values of the vertical deflections of each of these points the quantity δ occurring in the equation of three moments is at once derived.

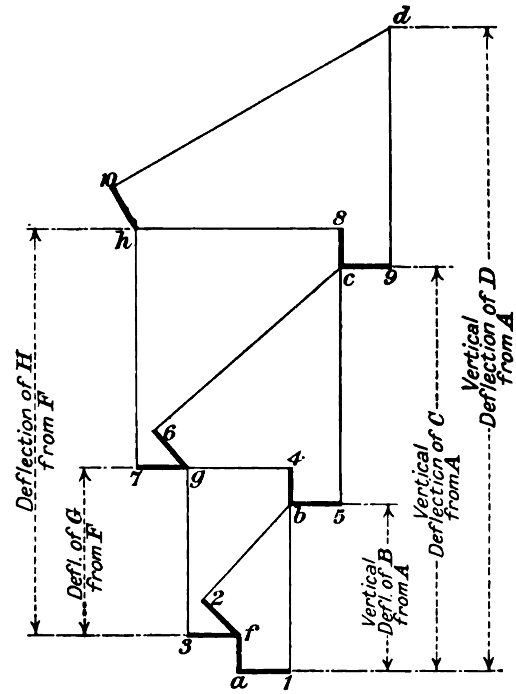


FIG. 90

§ **8.** *Alternative Method of calculating Deflections.* The graphical method of deriving the deflections given in the preceding paragraph may be supplemented by the alternative method of calculation given below.

Suppose the structure ABC, etc., fig. 89, initially unloaded, is subjected to a force L at B. By means of the triangle of forces the tensions and compressions which this originates (viz. in ABF) are obtained. Let the force in BF be mL where m is the force when L is unity. The structure will of course be deflected. Let

an external loading (for example, wind forces) be applied causing an additional deflection d of the structure at B. Let the corresponding extension in BF be e, the force in which member is now say $F + mL$. The work done by this extension of BF will be

$$\text{Average force in } BF \times \text{deflection in } BF = (\tfrac{1}{2}F + mL)\,e$$
$$= \tfrac{1}{2}Fe + mLe$$

The total work done under the final loading will then be the sum of these for all the members

$$= \Sigma\tfrac{1}{2}Fe + \Sigma mLe$$
$$= \text{work done by the wind forces} + L\Sigma me$$
$$= W + L\Sigma me$$

But the total work done may be regarded as composed of the work done by the extra wind loading + the work done when B under the load L is deflected an amount d to its final position $= W + Ld$. Equating these two statements and dividing by L we find

$$d = \Sigma me.$$

The deflection at B is thus the sum of the forces produced in each member by unit force at B, multiplied by the deflection in that member caused by the wind forces. Similarly for the other joints. The deflections having been calculated by either of the foregoing methods, the procedure is then identical with that already adopted in the case of the monoplane, and the corrected bending moments obtained. It may be noted in conclusion that in the calculation of the deflections sufficient accuracy may often be obtained, and the labour considerably reduced, by treating the wooden members as incompressible.

§ **9.** Taking F as the point of reference the deflections found by the first method given above are as in the table. The compressibility of the wooden members has been neglected and the area of cross-section of the wires has been taken as 0·085 sq. ins. in all cases. This area is that required to enable wire BF to withstand the tension of 4760 lbs. when the ultimate stress of the material is 50 tons/sq. in. and a factor of safety of 2 is used. The area of the wires must of course be modified to conform with standard wires obtainable from stock.

E is taken as 30×10^6 lbs./sq. in. in estimating extensions. Although the compressibility of the wooden members has been neglected, an approximate allowance for their effect could be

made by making a rough estimate of their cross-sectional area from the forces in Fig. 88. This has not been done, however, in the table below.

Joint	Deflection (inches)
A and F	zero
B and G	·372
C and H	·409
D	·426

In the foregoing discussion it has been assumed that the framework is pin-jointed at the junctions and consequently the next step in the analysis must be a determination of the extent to which the fact that the spars are in reality continuous members modifies the results. For this purpose it will be necessary to introduce at this stage a few comparatively elementary theorems on loaded beams required for use in the subsequent analysis. On these as a basis it will be possible to determine the extent of the modification introduced in the force distribution and to estimate to an approximation sufficient for practical purposes the bending moments along the spars.

§ **10.** Let the axes of reference be OX, OY, fig. 91, so that OY is parallel to the direction of lateral loading w and OX parallel to the end thrust F.

The following symbols will be used with reference to an element of the beam of length δx, deflections and forces being positive when acting as in figs. 91, 92 and 93:

x = abscissa in inches of any section of the beam measured from OY.

y = deflection in inches of that section measured vertically upwards from OX.

θ = slope of beam in radians.

w = lateral load in lbs. per inch run.

F = end thrust in lbs.

= force normal to a cross-section at x when parallel to OY.

f = shear in lbs. across this section.

T = thrust in lbs. normal to a cross-section at x when perpendicular to longitudinal neutral axis

= F approximately.

S = shear in lbs. across this section.
M = bending moment in lbs.-inches on section $b'c'$.
EI = flexural rigidity of beam at x in lbs. × sq. inches.
A = area of beam in sq. ins. at x.
R = radius of curvature of neutral axis.

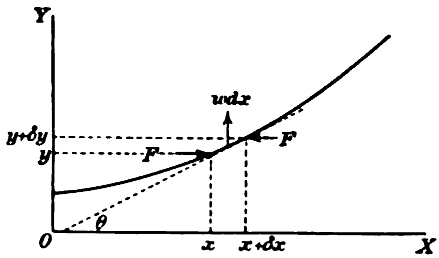


FIG. 91

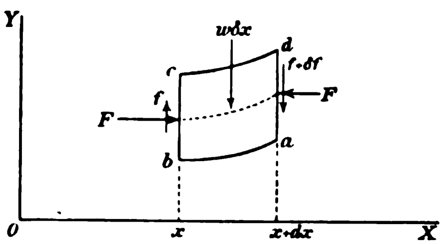


FIG. 92

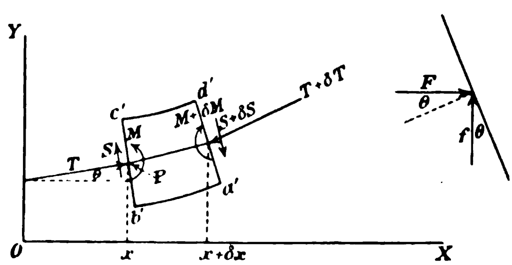


FIG. 93

Then it can easily be shown that:

(1) from fig. 91 by resolving forces on the element vertically

$$w = -\frac{df}{dx} \quad \text{..........................(1).}$$

(2) from fig. 93 by taking moments about P

$$-\frac{dM}{dx}=f-F\frac{dy}{dx}=S \quad \text{..................(2)},$$

since
$$S=f\cos\theta-F\sin\theta$$
$$=f-F\frac{dy}{dx}\text{ nearly,}$$

$$\therefore +\frac{d^2M}{dx^2}=+w+\frac{d}{dx}\left(F\frac{dy}{dx}\right)\text{ from (1)} \quad \text{.........(3)}$$

and by finding the extension or compression of elementary layers of the beam parallel to the neutral plane and assuming that plane sections remain plane we find on integrating that

$$\text{(3)}\quad M=-\left(E-\frac{F}{A}\right)\frac{I}{R}$$
$$=-EI\frac{d^2y}{dx^2}\text{ approximately} \quad \text{........................(4)},$$

since F/A is small compared with E and the curvature, approximately d^2y/dx^2, is negative when the moment tends to make the beam convex upwards.

§ **11.** The essence of the theory of the flexure of beams is contained in these three equations and nothing more than these will be utilized in the further development of the work of this section. The most convenient form of differential equation for the deflection y of the beam is found directly by inserting the expression found for M in (4) into equation (3) thus obtaining

$$\frac{d^2}{dx^2}\left(EI\frac{d^2y}{dx^2}\right)+\frac{d}{dx}\left(F\frac{dy}{dx}\right)+w=0 \quad \text{............(5)},$$

or since for the present we only treat those cases where F and EI are constant

$$EI\frac{d^4y}{dx^4}+F\frac{d^2y}{dx^2}+w=0 \quad \text{..................(6)}.$$

It will be noticed that the equation is of the fourth order, whereas it is usually given in the second order. The former can be obtained from the latter by differentiating twice, but equation (6) is in a form most suitable for directly inserting any type of end condition.

The solution of (6) when w represents uniform lateral loading is

$$y=A\sin\lambda x+B\cos\lambda x+Cx+D-\frac{wx^2}{2F} \quad \text{......(7)},$$

where $\lambda^2=F/EI$,

and A, B, C and D can be determined if the conditions at the end are known. This solution can easily be verified by differentiating and substituting in (7).

§ 12. *Theorem of Three Moments.* Suppose fig. 94 represents a part of a loaded beam resting on a number of supports of which three are indicated A, B and C. It will be justifiable to discuss the effect of the loading by concentrating attention on any portion of it provided the effect of the remainder of the beam on that portion is replaced by an equivalent shear and bending moment at the ends of the portion considered. It is proposed accordingly to find the effects of loading a series of separate beams AB, BC, etc., assumed initially unconnected, and then to make these continuous by the introduction of the requisite end conditions.

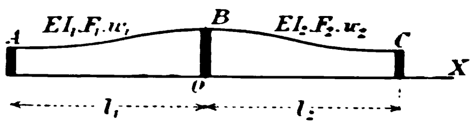


FIG. 94

Let EI_1 and EI_2 be the flexural rigidities, F_1 and F_2 the end thrusts, l_1 and l_2 the lengths and w_1 and w_2 the lateral loading of the two bays AB and BC respectively. Let d_0, d_1 and d_2 be the deflections and M_0, M_1 and M_2 the bending moments at A, B and C respectively.

Along BC

$$y_2 = U_2 \sin \lambda_2 x + V_2 \cos \lambda_2 x - W_2 x + P_2 - \frac{w_2 x^2}{2F_2} \quad \text{......(8)},$$

where $\lambda_2^2 = F_2/EI_2$.

Using $M = -EI\frac{d^2y}{dx^2}$ and the values for the deflections and bending moments at B and C, where $x = 0$ and $x = l_2$ respectively, we obtain, when these supports are simple supports, *i.e.* do not exert a moment on the beam,

$$\text{At } B \begin{cases} d_1 = V_2 + w_2, \\ \dfrac{M_1}{EI_2} = V_2\lambda_2^2 + \dfrac{w_2}{F_2}, \end{cases}$$

$$\text{At } C \begin{cases} d_2 = U_2 \sin\phi_2 + V_2\cos\phi_2 + W_2 l_2 + P_2 - \dfrac{w_2 l_2^2}{2F_2}, \\ \dfrac{M_2}{EI_2} = \lambda_2^2 (U_2 \sin\phi_2 + V_2\cos\phi_2) + \dfrac{w_2}{F_2}, \end{cases}$$

$$\therefore V_2 = \frac{M_1 l_2^2}{EI_2}\frac{1}{\phi_2^2} - \frac{w_2 l_2^4}{\phi_2^4 EI_2} \quad \text{.................................(9),}$$

$$P_2 = d_1 - \frac{M_1}{EI_2}\frac{l_2^2}{\phi_2^2} + \frac{w_2 l_2^4}{\phi_2^4 EI_2} \quad \text{...........................(10),}$$

$$W_2 = \frac{d_2 - d_1}{l_2} + \frac{w_2 l_2^2}{2\phi_2^2 EI_2} + \frac{1}{\phi_2^2}\{M_1 - M_2\}\frac{l_2}{EI_2} \quad \text{......(11),}$$

$$U_2 = \frac{1}{\phi_2^2 \sin\phi_2}\left\{M_2 - M_1\cos\phi_2 - \frac{w_2 l_2^2}{\phi_2^2}(1-\cos\phi_2)\right\}\frac{l_2^2}{EI_2}$$
$$\text{.........(12),}$$

where $\phi_2 = \lambda_2 l_2$.

Now the slope θ at B, where $x = 0$ in the beam BC, is

$$\begin{aligned}\theta_1 = \left(\frac{dy_2}{dx}\right)_{x=0} &= \lambda_2 U_2 + W_2 \\ &= \frac{1}{\phi_2 \sin\phi_2}\left\{M_2 - M_1\cos\phi_2 - \frac{w_2 l_2^2}{\phi_2^2}(1-\cos\phi_2)\right\}\frac{l_2}{EI_2} \\ &\qquad + \frac{d_2 - d_1}{l_2} + \frac{w_2}{2\phi_2^2}\frac{l_2^3}{EI_2} + \frac{1}{\phi_2^2}\{M_1 - M_2\}\frac{l_2}{EI_2} \\ &= \frac{M_1 l_2}{EI_2}\left(1 - \frac{\phi_2}{\tan\phi_2}\right)\frac{1}{\phi_2^2} + \frac{M_2 l_2}{EI_2}\left(\frac{\phi_2}{\sin\phi_2} - 1\right)\frac{1}{\phi_2^2} - \frac{d_1}{l_2} + \frac{d_2}{l_2} \\ &\qquad - \frac{w_2 l_2^3}{2EI_2}\left[\frac{\tan\phi_2/2}{\phi_2/2} - 1\right]\frac{1}{\phi_2^2} \quad \text{.....................(13).}\end{aligned}$$

Similarly the slope at B where $x = l_1$ in the bay AB is

$$\begin{aligned}\theta_1 &= \frac{M_0 l_1}{EI_1}\left(\frac{\phi_1}{\sin\phi_1} - 1\right)\frac{1}{\phi_1^2} - \frac{M_1 l_1}{EI_1}\left(1 - \frac{\phi_1}{\tan\phi_1}\right)\frac{1}{\phi_1^2} \\ &\qquad - \frac{d_0}{l_1} + \frac{d_1}{l_1} + \frac{w_1 l_1^3}{2EI_1}\left[\frac{\tan\phi_1/2}{\phi_1/2} - 1\right] \quad \text{..................(14).}\end{aligned}$$

Equating these two quantities we obtain

$$M_0 A_1 + M_1 (B_1 + B_2) + M_2 A_2$$
$$= -\frac{d_0}{l_1} + d_1 \left(\frac{1}{l_1} + \frac{1}{l_2}\right) + \frac{d_2}{l_2} + K_1 + K_2 \text{.........(15)},$$

where
$$A = \frac{l}{EI} \cdot \frac{1}{\phi^2}\left(\frac{\phi}{\sin\phi} - 1\right) = \frac{l}{EI} \cdot \alpha,$$
$$B = \frac{l}{EI} \cdot \frac{1}{\phi^2}\left(1 - \frac{\phi}{\tan\phi}\right) = \frac{l}{EI} \cdot \beta,$$
$$K = \frac{wl^3}{2EI}\left[\frac{\tan\phi/2}{\phi/2} - 1\right]\frac{1}{\phi^2} = \frac{wl^3}{2EI} \cdot \gamma.$$

§ **13.** The quantities α, β and γ are given in the following table for various values of ϕ when F is positive, *i.e.* F represents a compression, and by α', β' and γ' when F is negative. Generally we may write for the rth and $(r + 1)$th bays

$$M_{r-1} A_r + M_r (B_r + B_{r+1}) + M_{r+1} A_{r+1}$$
$$= -\frac{d_{r-1}}{l_r} + d_r \left(\frac{1}{l_r} + \frac{1}{l_{r+1}}\right) - \frac{d_{r+1}}{l_{r+1}} + K_r + K_{r+1} \text{......(16)}.$$

This provides an equation between the three bending moments at three neighbouring simple supports in terms of the lateral loading, length, flexural rigidity and end thrust of the two contiguous bays, and the deflections at the supports. When these quantities are known it is a simple matter to determine the bending moments at all the supports of such a system if the bending moments at two of the supports, usually the end supports, be known; for by writing down the equation of three moments for each set of three consecutive supports sufficient simple linear equations are obtained to solve for the required quantities.

The bending moments at intermediate points along the bays can then be found at once by differentiating equation (8) twice, and multiplying by $-EI$, inserting the values for U, V, W and P given in equations (9), (10) etc. and the values for the bending moments at the supports as already evaluated.

The quantities on the right-hand side of equation (16) and the coefficients A and B can be evaluated from the loading and the dimensions of the spars but it should be noted that in the preliminary design of a machine the latter dimensions are of course unknown, and some form of approximate estimate of this quantity would be necessary before the equation could be used in its

present form. This is most conveniently accomplished by neglecting the end thrusts and deflections at the supports and calculating the flexural rigidity and consequently the dimensions of the spar from the bending moments that would be derived under these conditions.

The various steps in the process will now be given for the framework under consideration.

$\phi = l\sqrt{\frac{F}{EI}}$ or $= l\sqrt{\frac{T}{EI}}$	$\alpha = \frac{1}{\phi^2}\left(\frac{\phi}{\sin\phi} - 1\right)$	$\beta = \frac{1}{\phi^2}\left(1 - \frac{\phi}{\tan\phi}\right)$	$\gamma = \frac{1}{\phi^2}\left(\frac{\tan\phi/2}{\phi/2} - 1\right)$	$\alpha' = \frac{1}{\phi^2}\left(1 - \frac{\phi}{\sinh\phi}\right)$	$\beta' = \frac{1}{\phi^2}\left(\frac{\phi}{\tanh\phi} - 1\right)$	$\gamma' = \frac{1}{\phi^2}\left(1 - \frac{\tanh\phi/2}{\phi/2}\right)$
0	·167	·333	·0833	·167	·333	·0833
·3	·167	·336	·086	·165	·331	·0826
·6	·175	·343	·088	·160	·325	·0805
·9	·186	·352	·091	·152	·317	·0774
1·2	·200	·367	·097	·142	·305	·0730
1·5	·222	·395	·109	·131	·293	·0679
1·8	·261	·436	·125	·120	·278	·0634
2·1	·323	·501	·148	·108	·264	·0581
2·4	·444	·630	·197	·097	·250	·0531
2·7	·711	·935	·310	·0872	·237	·0481
3·0	—	—	·843	·0777	·224	·0441
3·3	—	—	– ·770	·0696	·213	·0401
3·6	– ·684	– ·484	– ·262	·0620	·201	·0366
3·9	– ·440	– ·209	– ·153	·0553	·191	·0333
4·2	– ·331	– ·075	– ·100	·0495	·181	·0305
4·5	– ·278	·000	– ·071	·0444	·173	·0278
4·8	– ·252	+ ·066	– ·060	·0390	·165	·0256
5·1	– ·250	+ ·115	– ·051	·0360	·157	·0235

Near π, $\alpha = \frac{1}{\pi\theta}$

$$\beta = \frac{1}{\pi\theta} + \frac{2}{\pi^2}$$

$$\gamma = \frac{1}{\pi^2}\left(\frac{4}{\pi\theta} + \frac{12}{\pi^2} - 1\right)$$

when $\phi = \pi - \theta$.

§ **14.** *Evaluation of Bending Moments at Supports.* The two spars $A - E$ and $F - K$, fig. 87, can be treated separately. Neglecting the end thrusts and the deflections at the supports the equation of three moments becomes

$$M_{r-1} l_r + 2M_r (l_r + l_{r+1}) + M_{r+1} l_{r+1} = \tfrac{1}{4} (w_r l^3{}_r + w_{r+1} l^3{}_{r+1}).$$

Upper Spar.

Applying the equation to DCB we obtain since

$$M_D = 648 \text{ lbs. ins.}$$

$$648 \times 48 + 2M_C(48 + 72) + M_B \times 72 = 276000 + 1027000,$$

and for CBA

$$M_C \times 72 + 2M_B(72 + 72) + M_A \times 72 = 1027000 \times 2,$$

and for $B'AB$, where B' is the point on the other wing symmetrical to B,

$$M_B \times 72 + 2M_A(72 + 72) + M_B \times 72 = 1027000 \times 2.$$

These equations give:

$$M_A = 4620 \text{ lbs. ins.}$$
$$M_B = 5030 \quad ,, \quad ,,$$
$$M_C = 3790 \quad ,, \quad ,,$$

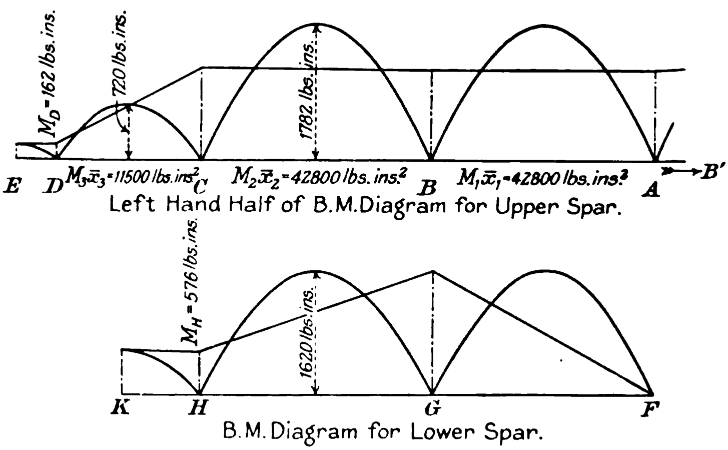


FIG. 95

Fig. 95 represents the bending moment diagram when the end thrusts, etc. are neglected.

Lower Spar.

In this case the spar is pin-jointed at F and therefore $M_F = 0$ and M_H is known, viz. $M_H = 2300$ lbs. ins. The unknown bending moment M_G can thus be obtained from

$$2300 \times 72 + 2M_G(72 + 72) = 936000 \times 2$$

giving $$M_G = 5900 \text{ lbs. ins.}$$

The bending moment diagram for the lower spar is given in Fig. 95.

§ **15.** *Calculation of Size of Spar, etc.* The breaking stress for spruce is roughly 8500 lbs./sq. in. In the case given above the loading is assumed four times normal and the factor of safety will be taken as 2 throughout. Since the thrust will increase the bending moments considerably, the factor of safety in the approximation will be taken as 3·6 ins.

The depth of the section is practically fixed by the thickness of the wing section at the position where the spar is to be placed. This thickness will be taken as 3·6 ins.

$$\therefore \frac{5030}{1{\cdot}9 \times (1{\cdot}6)^2} + 5760 = 3400A,$$

$$\therefore A = 2 \text{ sq. ins.},$$

giving a breadth of about $1\frac{3}{4}''$ and $I = 5{\cdot}1$ (inches)^4.

Taking E for wood as $1{\cdot}5 \times 10^6$, the flexural rigidity becomes $7{\cdot}65 \times 10^6$ lbs. (ins.)^2.

§ **16.** *Effect of End Thrust upon the Bending Moments.* We must now use the general equation of three moments, which is, when reduced to the form most suitable for our purpose,

$$M_{r-1}\, l_r . 6\alpha_r + 2M_r\,(l_r . 3\beta_r + l_{r+1} . 3\beta_{r+1}) + M_{r+1}\, l_{r+1} . 6\alpha_{r+1}$$
$$= EI\left\{\frac{-d_{r-1}}{l_r} + d_r\left(\frac{1}{l_r} + \frac{1}{l_{r+1}}\right) - \frac{d_{r+1}}{l_{r+1}}\right\} + \frac{w_r l_r^3}{4} . 12\gamma_r$$
$$+ \frac{w_{r+1} l_{r+1}}{4} . 12\gamma_{r+1} \qquad \text{..........(17).}$$

In the following table are given the quantities ϕ, α, β and γ required for equation (17) in effecting the next approximation to the bending moments.

Bay	$\phi = l\sqrt{\frac{F}{EI}}$	6α	3β	12γ
AB	1·97	1·76	1·41	1.64
BC	1·20	1·20	1·10	1·18
CD	0·29	1·00	1·00	1·00
		$6\alpha'$	$3\beta'$	$12\gamma'$
FG	1·20	0·85	0·91	0·88
GH	0·44	0·98	0·99	0·98

The equations for the bending moments at the supports using the tables for α, β, γ etc. now take the form:

Upper Spar.

$$648 \times 48 + 2M_C(48 \times 1{\cdot}00 + 72 \times 1{\cdot}10) + M_B \times 72 \times 1{\cdot}20$$
$$= 7{\cdot}65 \times 10^6 \left\{ -\frac{0{\cdot}426}{48} + 0{\cdot}409\left(\frac{1}{48} + \frac{1}{72}\right) - \frac{0{\cdot}272}{72} \right\}$$
$$+ 276000 \times 1{\cdot}00 + 1027000 \times 1{\cdot}18,$$

$$M_C \times 72 \times 1{\cdot}20 + 2M_B(72 \times 1{\cdot}10 + 72 \times 1{\cdot}41)$$
$$+ M_A \times 72 \times 1{\cdot}76$$
$$= 7{\cdot}65 \times 10^6 \left\{ -\frac{0{\cdot}409}{72} + 0{\cdot}272\left(\frac{1}{72} + \frac{1}{72}\right) \right\}$$
$$+ 1027000 \times 1{\cdot}18 + 1027000 \times 1{\cdot}64,$$

and

$$M_B \times 72 \times 1{\cdot}76 + 2M_A \times 2 \times 72 \times 1{\cdot}41 + M_B \times 72 \times 1{\cdot}76$$
$$= 7{\cdot}65 \times 10^6 \left\{ -\frac{0{\cdot}272}{72} \times 2 \right\} + 1027000 \times 1{\cdot}64 \times 2$$

giving $M_A = 4800$ lbs. ins.

$M_B = 5440$,, ,,

$M_C = 3950$,, ,,

The deflection terms, it will be noticed, have very little influence upon the result and could be neglected for the upper spar.

Lower Spar.

$$2300 \times 72 \times 0{\cdot}98 + 2M_G(72 \times 0{\cdot}99 + 72 \times 0{\cdot}91)$$
$$= 7{\cdot}65 \times 10^6 \left\{ -\frac{0{\cdot}409}{72} + 0{\cdot}272\left(\frac{1}{72} + \frac{1}{72}\right) \right\} + 936000(0{\cdot}98 + 0{\cdot}88)$$

giving $M_G = 5820$ lbs. ins.

In this case the deflections play only a small part and might be neglected in most cases.

Using these values for M_A, M_B, M_C, and M_G the bending moment curves may be corrected as previously explained and the spars re-designed. Comparatively little alteration as can be seen has resulted from the inclusion of the end thrust etc., but this is not necessarily the case generally.

The next step in the process must clearly be to determine the reaction at the supports and then recalculate the force distribution. The reaction will evidently modify the values of F_1

and F_2 previously used. Although this will probably have little effect on the values of the bending moments as already obtained, the stress due to direct thrust might be altered to a measurable extent in some cases.

§ **17.** *Reaction at the Supports.* Consider [fig. 96] an element of the beam at a support. Let f_l be the vertical shear to the left of this element and f_r that to the right. Since this element is in equilibrium the reaction R is clearly given by

$$R = f_r - f_l \quad \text{..........................(18).}$$

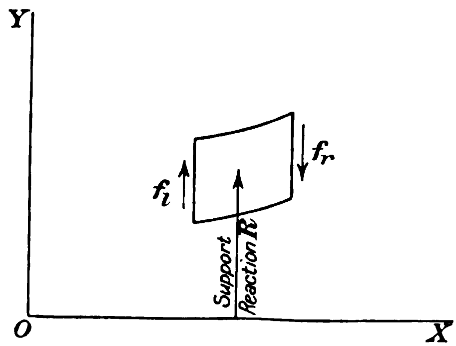


FIG. 96

Taking for example support B, fig. 94, then from (2)

$$f_r = S + F_2 \frac{dy}{dx}$$

when x is put equal to zero for the second bay BC.

Now $$S = -\frac{dM}{dx} = EI_2 \frac{d^3y}{dx^3},$$

$$\therefore f_r = EI_2 \frac{d^3y}{dx^3} + F_2 \frac{dy}{dx} \quad \text{..............(19),}$$

when $x = 0$. From (19) using (8) at $x = 0$

$$f_r = \frac{1}{l_2}\left\{M_1 - M_2 + \frac{w_2 l_2^2}{2} - F_2(d_1 - d_2)\right\} \quad \text{......(20).}$$

Similarly for the bay AB

$$f_l = -\frac{1}{l_1}\left\{M_1 - M_0 + \frac{w_1 l_1^2}{2} - F_1(d_1 - d_0)\right\} \quad \text{......(21),}$$

giving

$$R_2 = -\frac{1}{l_1}\{M_0 - M_1 - F_1(d_0 - d_1)\} + \frac{w_1 l_1}{2}$$
$$+\frac{1}{l_2}\{M_1 - M_2 - F_2(d_1 - d_2)\} + \frac{w_2 l_2}{2} \dots\dots\dots(22).$$

But originally the reaction was taken as $\frac{1}{2}(w_1 l_1 + w_2 l_2)$ and thus the effect of the continuity of the spar, etc. increases the upload at the joints by an amount given by

$$\tfrac{1}{2}\{w_1 l_1 + w_2 l_2\} - R_2 = R_2' \text{ say}$$
$$= \frac{1}{l_1}\{M_0 - M_1 - F_1(d_0 - d_1)\} - \frac{1}{l_2}\{M_1 - M_2 - F_2(d_1 - d_2)\}.$$

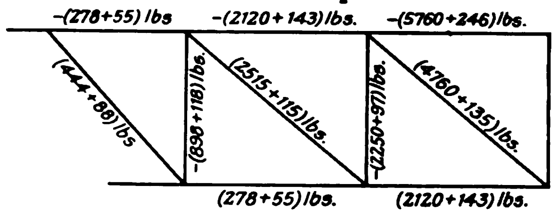


FIG. 97

The increased loads at A, B, C etc. (fig. 88) will therefore become:

Upper Spar.

$$R^{D'} = -\frac{1}{48}\{658 - 3950 - 278 \times 0{\cdot}017\} = 69 \text{ lbs.}$$

$$R_C' = -69 - \frac{1}{72}\{3950 - 5400 - 2120 \times 0{\cdot}137\} = -44{\cdot}8 \text{ lbs.}$$

$$R_B' - -24{\cdot}2 - \frac{1}{72}\{5400 - 4800 - 5760 \times 0{\cdot}272\} = -10{\cdot}7 \text{ lbs.}$$

$$R_A' = -2\{R_B' + R_C' + R_D'\} = -6{\cdot}75 \text{ lbs.}$$

Lower Spar.

$$R_H' = -\frac{1}{72}\{2300 - 5820 - 278 \times 0{\cdot}137\} = -49{\cdot}4 \text{ lbs.}$$

$$R_G' = -49{\cdot}4 - \frac{1}{72}\{5820 - 2120 \times 0{\cdot}272\} = -23{\cdot}4 \text{ lbs.}$$

$$R_F' = -(R_H' + R_G') = -72{\cdot}8 \text{ lbs.}$$

The deflections it will be noticed affect the reactions to a much more considerable extent than they did the bending moments. From the value of R_A' etc. given above the forces in the members can now be re-calculated. These are given in fig. 97.

The estimate of the forces in the members has been affected by approximately 10 per cent., an amount well worth considering in the final calculation of the size of the members. This increase, however, modifying but slightly the end thrust F, used in the equation of three moments, and the deflections, there will be no necessity to recalculate the bending moments. The bending moment diagram when corrected for end thrust determined by the original force distribution together with the forces set out in fig. 97 may be taken as sufficiently accurate for determining the stresses in the members.

If the members be now finally modified in dimensions to suit the requirements of these stresses, very little modification will be involved in the foregoing calculation by these changes.

§ **18.** The principles utilized in this analysis must be clearly borne in mind. The biplane framework is in the first instance assumed composed of just sufficient members to make it rigid on the assumption that all the junctions present are pin-jointed. The distribution in force calculated on this basis is then used as a starting point and first approximation for a new framework in which the spars are no longer regarded as composed of pin-jointed sections but of continuous members.

It is clear that the continuity of these parts operates towards strength in what is otherwise a rigid framework. Such a factor is usually referred to as a *redundancy*. The fuselage it will shortly be seen is an example of a structure essentially involving redundancies.

Apart from those already encountered in the wing structure itself, an attempt to face the question of the effect of longitudinal loading on the wing spar in the plane of the wing immediately brings into consideration the modifying effect of incidence wires, themselves of a redundant nature.

§ **19.** *Rigidity and Redundancy of frameworks.* The difficulty of determining the force distribution in a structure, such as a fuselage, for example, under a given loading lies essentially in the practical difficulty of isolating the effects of the components into which the loading may be conveniently analysed. In general, a fuselage, supposed fastened to fixed supports, which the frame of the engine may be considered to be, may be regarded as subject

to forces and couples applied to a particular member or members as a result of the aerodynamic loading. This combination of forces and couples can be resolved along three axes, and if it be known which ties and members of the structure are eventually effective in bearing the stress, then, and only then, can the several components of the loading be treated separately, and their effects superposed. That this is so, is apparent from the simple consideration of the cross-bracing of a rectangular frame in which the calculated distribution of the forces in the members will be widely different according to which of the diagonal members are assumed operative. In the case of the fuselage an axial torsion at the tail will eliminate certain bracing wires and give rise to a definitely calculable force distribution in the resulting geometric frame. A simple down-load on the tail, however, would reintroduce some of these members and eliminate others. The algebraic sum of the two-force distributions thus calculated on what are essentially two widely different frameworks would not be even approximately the same as that originated when both torsion and download, say, operate simultaneously.

When experimental work which would provide an insight into the physical assumptions and simplifications that might be introduced is not available, mathematical analysis may provide useful information if certain difficulties could be overcome. Principal among these is that involved in the question of redundancies which owe their existence, of course, primarily, for supplying stiffness to the elastic structure. Mathematically the difficulty of treatment, where such members occur, lies essentially in the fact that they are not always operative, and that generally it is not possible to predetermine the conditions under which a particular member may go out of action. This will depend entirely on the extensions that arise in the material of the structure, and is consequently intimately associated with the manner in which the load is taken up.

§ **20.** Since a structure may in general be regarded as simply a combination of members (ties and struts) and joints (ball and socket in the case of three dimensions and pin-joints in the case of two), a rigid, non-redundant structure will be one in which the distances apart of the joints given in terms of the struts and ties are just sufficient to determine uniquely their relative positions.

In the special case in which three pin-joints in a plane frame, for example, are on the same member the latter may be regarded, as far as the geometrical specification is concerned, as the limit of a triangle where two sides coincide with the third. If the frame is under-specified, so that one or more of the joints has a degree of freedom, the structure is non-rigid, while if it is over-specified so that one or more of the members could be removed and yet the relative geometrical positions of the joints remain unaltered, the structure is redundant. Generally speaking, structures cannot be absolutely classified into these three categories, since it is evidently possible to have a structure of which the joints in one portion are over-specified and those in another under-specified, so that the resulting framework is partly redundant and partly non-rigid.

Consider the case in two dimensions in which there are p joints with one of the members assumed as held rigid. It is proposed to investigate what conditions must be satisfied in order that the structure be uniquely determined and not over-specified. Let a particular member be one of two axes x and y. There are then $2(p-2)$ co-ordinates to be found for the remaining $p-2$ joints. These must be specified by the lengths of the remaining $(m-1)$ members of the framework, providing $m-1$ conditions. Hence the necessary relation that must be satisfied is

$$m - 1 = 2p - 4,$$

or

$$m = 2p - 3.$$

This is usually obtained on the quite illegitimate assumption that the framework is composed of triangles.

It should be remarked that the satisfaction of this equation is evidently not sufficient, although it is commonly considered so, since, as has already been explained, the $m-1$ lengths might be such as to over-specify quite consistently a certain number of the joints and to under-specify to the same extent the remainder. In such an event the framework will be partly redundant and partly non-rigid. It can only be definitely stated in general that if this equation indicates that the structure is under-specified, then a certain portion at least of the framework must be non-rigid; if it indicates the structure is over-specified, then a certain portion at least must contain redundant members. If it is satisfied no definite conclusion can rigorously be drawn, although the converse

is perfectly sound that every rigid non-redundant framework must satisfy this equation.

A simple illustration that serves to throw further light on these points is provided by the frame shown in fig. 98 *a*, where AD, BE and CF are parallel and AC and DF continuous at B and E, or the more general frame fig. 98 *b*. The equation

$$m = 2p - 3$$

is clearly satisfied in both cases, but it is equally evident that the frames are not rigid, but the forces in the members when the frame is loaded will become infinitely large unless the parallelism of AD, BE and CF be destroyed. The reason for this is simple when it is noted that as long as BE and CF are retained parallel and the lengths of AC and DF such as to keep AD parallel to BE, the specification of the length of AD is unnecessary, since it can be derived from the lengths of the other members.

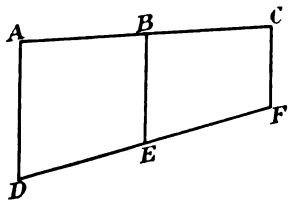


FIG. 98 *a*

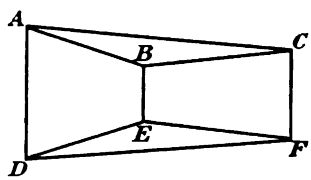


FIG. 98 *b*

This illustration is of particular interest in view of the fact that similar frameworks may arise in three dimensions, as in the case of a fuselage.

In three dimensions the corresponding equation with its equivalent restrictions can similarly be shown to be

$$m = 3p - 6.$$

The interpretation of these equations from the point of view of the determination of the forces in the members is worthy of consideration. If p be the number of joints in three dimensions, then resolving the forces at each joint along three co-ordinates, $3p$ equations will be obtained. From equilibrium among the externally applied forces there are furnished three directional equations and three stating that the couples produced by these forces are zero. There are thus $3p - 6$ independent equations among the tensions and compressions in the m members. For consistency it follows that

$$m = 3p - 6.$$

This indicates that the definition of a rigid, non-redundant frame is essentially one in which the forces can be obtained by direct resolution, but it is important to remark that, when certain relations are satisfied, the system of linear equations determining the force distribution under a special type of loading for an otherwise rigid, non-redundant frame might contain two or more equations which are identical, and the distribution would thus be indeterminate. In a sense, therefore, the framework will be redundant to a particular type of loading and would require to be treated as such.

On the other hand, if the structure shown in fig. 98 *b* be subjected to loading (where *AD* and *CF* are parallel and *BE* just not parallel to these), and if the structure be elastic so that under loading *AD*, *BE* and *CF* may be deflected and become parallel, the whole framework suddenly assumes the nature of a mechanism and collapses. The framework is in fact unstable.

§ **21.** In view of the fact that these equations are not in themselves a sufficient criterion for determining when a structure is rigid or otherwise, and in view of the necessity for this determination before stress calculations can be entered into, further tests must be considered. Thus from an analysis of the framework the redundant members must be fixed upon, supposed removed and replaced by equivalent externally applied forces acting upon the remainder of the frame now rigid and non-redundant. The analysis would normally proceed according to the principle of strain energy or a method to be given later in this chapter, but in general it is extremely difficult to specify where exactly the redundancy lies, and accordingly what is to be the nature of the rigid framework to which the structure is to be reduced. A number of methods might be adopted to simplify a complicated framework. For example, it is clear that initially all joints in a plane frame at which only two members meet may be supposed removed together with the two members in question, and the resulting frame will not be affected as regards the question at issue. In the same way in three dimensions all points at which three, and only three, members meet may likewise be removed. In the second place, if in any structure a series of members and joints may be so grouped together as in themselves to form a rigid non-redundant frame, this latter may be supposed removed and

replaced in the case of a plane frame by a single member with joints at the ends. This may also be done at intermediate points where the member itself is continuous or is attached to the remainder of the structure. This is legitimate in view of the fact that such a self-contained constituent portion is rigid in itself, and therefore is in essence equivalent to one member. The fact that the rest of the structure may be joined to this equivalent member at points other than the two ends does not vitiate the conclusions, since fixing the self-contained portion in space involves also fixing the position of these joints.

In certain cases it will be possible to transfer all intermediate joints to the ends, but care must be exercised to avoid the introduction of an extra degree of freedom to the remainder of the structure. This often arises when the transformation involves the coalescence of two members. In a like manner if any portion of the structure in three dimensions is rigid and not redundant in itself it may be replaced by a triangular plate to which are attached the various members originally connecting this fixed portion to the remainder of the structure. The resulting intermediate joints may then be transferred to any of the apices of the plate, provided in so doing no extra degree of freedom is given to the latter, a condition which can easily be tested in any particular case. It may not always be possible, of course, to find such a triangle or tetrahedron from which to start off, but in most cases it is the simplest and most convenient method of reduction.

In this connection it is worthy of note that it is advisable in fixing the rigid, non-redundant frame from which to commence the analysis to ensure that the rigidity does not depend on the fact already brought out in figs. 98 *a* and 98 *b*. If the frame be rigid because certain members are just not parallel, an analysis of the distribution in force will indicate that these become excessively large, and the results will consequently be very inexact.

§ **22.** Even when the framework has been thus reduced and determined to be a non-redundant structure, it is not always a simple matter even in the comparatively elementary case of a two-dimensional framework to determine the distribution. It is normally supposed that for such cases an application of Bow's notation will suffice, but it can easily be shown that whole classes of quite simple two-dimensional frameworks exist, non-redundant

and rigid, for which Bow's notation will fail to determine the force distribution for external loading. In fact, the method is only really applicable in those very special cases in which one joint at least exists in the framework at which there are only two members, and not always even then. Only a very limited class of two-dimensional rigid and non-redundant frameworks are, however, of this nature, for it can easily be shown that a framework may have as many as three members at every joint under certain conditions to be explained shortly, and yet the frame will not be over-specified. It would then be impossible to apply Bow's notation directly, although by an artifice the frame can be so modified as to make it amenable to that method. Consider in illustration (fig. 99) a modified form of fig. 98 *b*, in which for simplicity *ABCD* is a square and the angles are as specified. There are six joints and nine members, so that the equation is satisfied. By the method of reduction already indicated it can easily be shown that the frame is rigid and non-redundant, but three members radiate from each point, so that Bow's notation cannot be applied directly. Let the external forces W act along a diagonal. Then, considering the joint A and replacing the member AB by the two externally applied forces P at A and B, we find by resolution along AB,

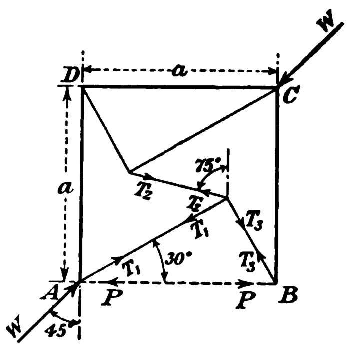


FIG. 99

$$P = T_1 \frac{\sqrt{3}}{2} + \frac{W}{\sqrt{2}}$$

at the joint E by resolution horizontally and vertically

$$T_3 = -T_1,$$

and at the point B

$$P - T_3/2 = 0.$$

Eliminating T_1 and T_3 we easily find

$$P = \frac{W}{(1+\sqrt{3})\sqrt{2}}.$$

The force P being thus known in terms of W, it may be considered completely as an external force, and the forces in the

remaining members determined by a direct application of Bow's notation. In the present case, however, on account of the symmetry of the loading and of the frame this is unnecessary, but the artifice is of quite general application; it is, in fact, only necessary to treat one member as an external force at one end and work round to the other by the shortest route.

§ 23. The generality of this artifice is not limited to two dimensions. It is a simple matter to show that rigid, non-redundant frames can be constructed in three dimensions, from every joint of which four members radiate, with six joints in all. Under these circumstances, one of the members may likewise be replaced by an equivalent external force, and the analysis carried through by resolution of forces. There is, of course, no extended form of Bow's notation for such a case, so that the method simply reduces here to direct resolution, but it is impossible, as is normally done, to proceed from joint to joint, determining the forces step by step.

§ 24. Interesting cases of rigid frameworks, at every joint of which at least four members meet, frequently arise in the simplification of aeroplane frameworks by the method of reduction already explained. A case in point is that shown in fig. 100, where the quadrilateral $ABCD$, not necessarily plane, has each of its four vertices joined to the points P and P'. It would appear at first sight that the structure would collapse under compression in the direction BD or AC, but this is not so. By inspection it is seen that the equation is satisfied and that four members radiate from each point. Generally, in fact, any polygon of n sides not necessarily co-planar, every joint of which is connected directly to external points P and P', will form a rigid frame with $3n$ sides and $n + 2$ points, thus satisfying the equation. The series of frameworks included in this class will be found extremely useful in applying the method of reduction.

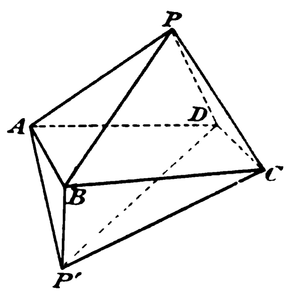


FIG. 100

It may be remarked in passing that structures which have five or more members at each joint must possess at least twelve joints at which only five members meet.

§ 25. It may be stated then generally as a conclusion from the foregoing discussion that any rigid non-redundant structure in three dimensions under external loading can be analysed as far as force distribution is concerned by the direct process of resolution of forces at each joint. A system of linear equations will be obtained which can, of course, be solved. The analysis from this point of view, in fact, may be used as a criterion when other methods fail for determining whether the structure is under- or over-specified. Unfortunately, this method of solution does not commend itself to the drawing office, as no general graphical method appears to exist for three-dimensional problems, but in many cases of considerable importance it is possible to regard certain three-dimensional frameworks—aeroplane fuselages and wing structures in particular—as composed of plane or nearly plane rigid structures fastened together, and to treat these separately, due allowance being made for the effect of the junctions between the component plane portions and for the small curvatures in the latter. The method, it will be seen, will apply in all cases in which the structure may be developed into a series of separate plane frameworks, provided it is possible to determine by some means which components of the external loading are taken up by the constituent developed elements of the structure. In the case of a fuselage, where this method of development can easily be carried through, this is no difficult matter for any type of external loading at the tail. The investigation will at the same time give an indication of the degree of approximation obtained by assuming the sides of the fuselage plane.

§ 26. *Analysis of Stresses in a Fuselage.* For the present purpose it will be assumed that the fuselage supposed inelastic is of the type shown in fig. 101, neglecting for the present modifications due to position of cockpit, etc., although the method will be applicable generally. The framework consists essentially of four sides:

Front face;	top edge	a_1, b_1, c_1, d_1	...	m_1
	bottom edge	a, b, c, d ...	...	m
Top face;	front edge	a_1, b_1, c_1, d_1	...	m_1
	rear edge	a_1', b_1', c_1', d_1'	...	m_1'

Rear face;	**top edge**	a_1', b_1'	... ...	m_1'
	bottom edge	a', b'	... ...	m'
Bottom face;	**front edge**	a, b, c	... ...	m
	rear edge	a', b', c'	... ...	m'

Size of Members (inches).

$aa_1 = 11{\cdot}1$	$aa' = 0$	$a_1b_1 = 19{\cdot}75$
$ff_1 = 27{\cdot}9$	$bb' = 11$	$b_1c_1 = 22{\cdot}5$
$gg_1 = 30$	$cc' = 15$	$c_1d_1 = 22{\cdot}5$
$hh_1 = 30$	$gg' = 23$	$d_1e_1 = 22{\cdot}5$
$kk_1 = 30$	$mm' = 23$	$e_1f_1 = 22{\cdot}5$
$mm_1 = 28{\cdot}5$		$f_1g_1 = 22{\cdot}5$
		$g_1h_1 = 18$
		$h_1k_1 = 17{\cdot}5$
		$k_1m_1 = 21{\cdot}75$

In elevation the longerons are straight between aa_1 and ff_1, and also between gg_1 and kk_1.

In plan the longerons are straight between cc_1 and gg_1, and between gg_1 and mm_1.

The fuselage tapers to the edge aa_1 (a and a', a_1 and a_1' are respectively identical). The points m, m_1, m', m_1' will be considered as attached to the framework of the engine, which itself will be regarded as a rigid structure, so that the four points k, k_1, k', k_1' will then be uniquely fixed and not over-specified. For the moment everything to the right of the bulkhead is excluded at k, the cross-bracing is of the type shown in fig. 101, provided only one of the bracing wires (chain dotted) is operative, and k, k_1, mk, etc., are rigid members with ball and socket joints at m, k, etc.

Hence ultimately the four points b, b_1, b', b_1' will also be uniquely fixed, but this would give rise to one redundancy at the rudder post unless it be supposed that the latter is replaced by a rectangular frame a, a_1, a', a_1', where aa' and a_1a_1' are very small and jointed at these points to their respective longerons, abc ... $a_1b_1c_1$ ... $a'b'c'$ ... and $a_1'b_1'c_1'$. A diagonal wire or strut inserted across this frame will make the conditions identical with those arising from the existence of the rudder post, and this redundancy must be specially treated by the ordinary strain energy or an equivalent method to be considered later. The cross wires in the last compartment on the upper and lower faces may be supposed to coalesce with the longerons.

§ 27. In the general case the aerodynamic loading (neglecting gap struts and the connection to the undercarriage) may be

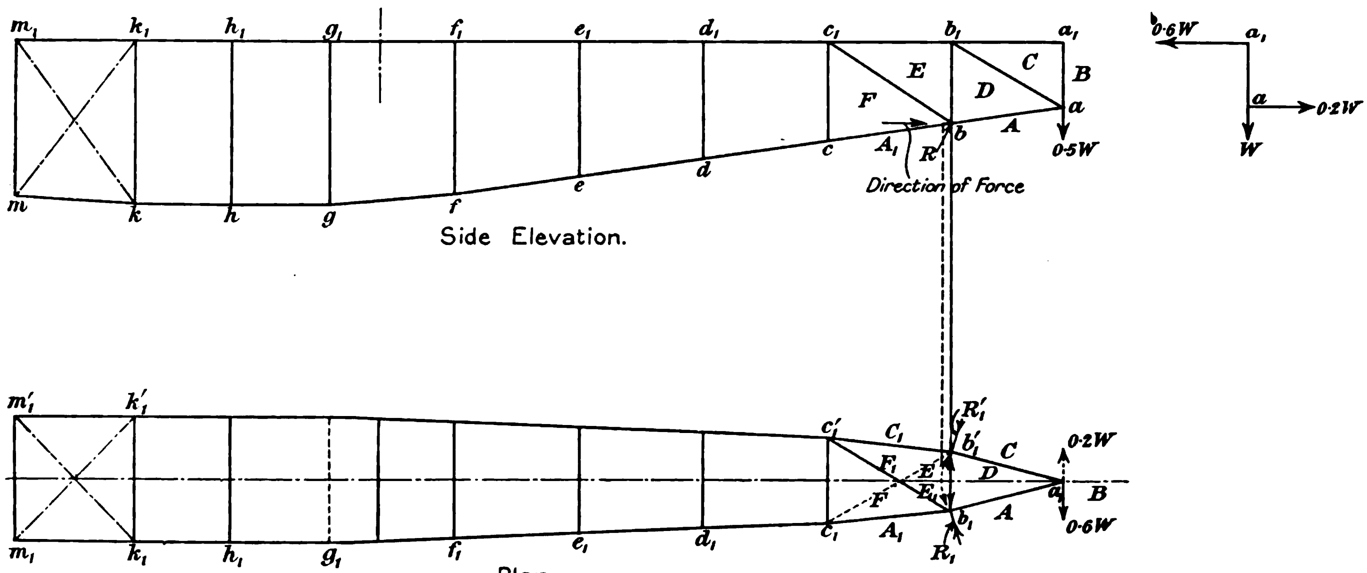


FIG. 101. Stresses in loaded rigid structures with redundancies.

conveniently reproduced by the application of external forces at the joints. For the present, however, there will merely be considered forces due to the rudder and fin, tail plane and elevators. Assuming no redundancies in the type of loading, this may be conveniently represented as consisting of forces of $0{\cdot}5W$, say, at a on ab and a' on $a'b'$ acting directly downwards, due to tail plane and elevators, and lateral forces of $0{\cdot}6W$ and $0{\cdot}2W$ in opposite directions along a_1a_1' and aa' respectively as in fig. 101, originated by the lateral force on the rudder. For complete generality in loading there ought strictly to be included other forces and moments also—for example, moments about aa_1 and about an axis perpendicular to the plane of symmetry of the machine. These are not taken into consideration, as their magnitudes, depending partially upon the manner of connection between the tail system and fuselage, are very variable.

The fuselage may be regarded as consisting of four developed and entirely separate sides with their respective cross-bracings, struts and longitudinals.

In the actual fuselage considerable trouble must arise in the neighbourhood of the rudder post, on account of the nature of the redundancy present. From an analysis of this must be determined the precise manner in which the external loading at the tail will be taken up by the constituent developed faces. Where the redundancy actually exists can be seen a consideration of the structure at the tail. (Fig. 102.)

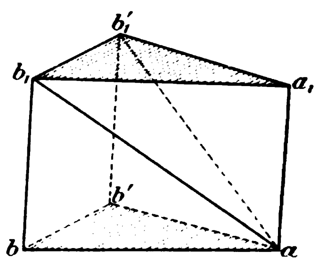


FIG. 102

The four points b, b', b_1, b_1' are fixed by the remainder of the structure between that bulkhead and the engine frame. The tetrahedron $a_1ab_1b_1'$ may be removed from the structure, being self-contained and rigid, and there remains only the pyramid $abb_1b_1'b'$. The last four points being fixed, a also will be fixed by only three of ab, ab_1, ab_1', ab', the remaining one being redundant.

If there were merely a simple download W on the tail, it is obvious from symmetry that it would be distributed evenly between the two vertical faces even when the redundancy is

present, but the situation is otherwise when a lateral force or a twisting couple is also present. In this case the component loadings on the faces may be considered as $\frac{w}{2} + P$ and $\frac{w}{2} - P$ where P is determined by the nature of the redundancy, and must be specially treated as such. Even when the rudder post is regarded as the limit of a very narrow rectangle with cross-bracing the same consideration applies, the redundancy appearing as a compression or a tension in the rudder post. If the faces of the fuselage are plane and there are no cross-bracings in the bulkheads, then w is zero and the download is equally distributed.

For the present, however, only the non-redundant frame will be considered, and it will then be accurately true to suppose that the loading is taken up, so that the force $0{\cdot}5W$ acts on ab and is distributed in the vertical face containing this member according to the method shortly to be outlined. Thus this vertical face is supposed developed into a plane surface and treated as a simple framework with the points m and m_1 rigidly fixed. In the same way the loads $0{\cdot}5W$ on $a'b'$, $0{\cdot}6W$ along a_1a_1', and $0{\cdot}2W$ along aa' may be considered as acting upon the respective faces developed into the corresponding plane frameworks. In normal practice this is, of course, the method usually adopted where the forces in common members are added algebraically.

The modification of this method that will now be adopted here will furnish the forces in the members in the fuselage under the given conditions of non-elasticity and non-redundancy exactly. but it will be seen that the results of the foregoing method provide a fairly good approximation to the truth when the frame is modified, as stated above.

§ **28.** Consider the four faces developed into four distinct plane pieces, the common longerons supposed split in such a manner as to leave the stresses in corresponding members equal. This is no assumption, of course, but only a convenient method of regarding the question, since the cross-sections of the longerons do not enter as a factor in the problem at all. It is proposed gradually to bring these frameworks together by making them step-by-step coincident from one end with the original fuselage, and to estimate the constraints that must be imposed in order to maintain these developed frameworks in their undeveloped

positions. The next step is then clearly to discover how these constraining forces are taken up by the various members of the structure. Let then the four developed faces be brought together to coincide as they would in the original structure along aa_1, $a'a_1'$, a_1a_1', ab, a_1b_1, $a'b'$, $a_1'b_1'$, but otherwise plane. The points m, m_1, etc., at the extreme ends must for the moment be assumed fixed in space in their new positions. There is, of course, so far no connection between the four faces except at the rudder post at which the loads are applied. Under these circumstances, the force distribution will be identical with that already derived by Bow's notation. Clearly the two portions of the longerons into which a_1b_1 was divided may be bound together without the introduction of any constraints, and the force distribution to the right of the first bulkhead is now definite and cannot be altered by any modifications to the left of that bulkhead. Thus the tensive force in the wire ab_1, fig. 101, is given by CD, fig. 103 a, the force in ab is given by the algebraic sum of the forces (in this case the absolute differences) represented by AD, fig. 103 a, and AD, fig. 103 d, and similarly for the other members to the right of the bulkhead bb_1.

Now suppose the vertical sides bent round, about the positions bb_1, $b'b_1'$ respectively, so that the constituent elements of each of the longerons b_1c_1, $b_1'c_1'$, bc, $b'c'$ coincide. To maintain this position certain constraints must necessarily be introduced, and it remains to estimate these. Consider the constraint at b. To maintain the force distribution in the front face a constraint equal and opposite to R, fig. 101, obtained by finding the resultant of the forces in the three members ba, bc, bc_1 in the bend position must be applied. R is clearly a force in a horizontal plane and provides a measure of the additional force that the whole frame to the left of the bulkhead bb_1 must take up. In the same way the forces R_1, R', R_1' which the frame must take up at the other corners in order to maintain the fuselage structure to the right of the bulkhead cc_1 can be derived. These forces must now be treated as external loads for the remainder of the undeveloped structure, and the resulting force distribution once more derived if necessary by Bow's notation (see fig. 103). Again if the areas of the common portions of the longerons are assumed so divided that the stresses are equal, the members up to the bulkhead cc_1 may be bound together without the introduction of further con-

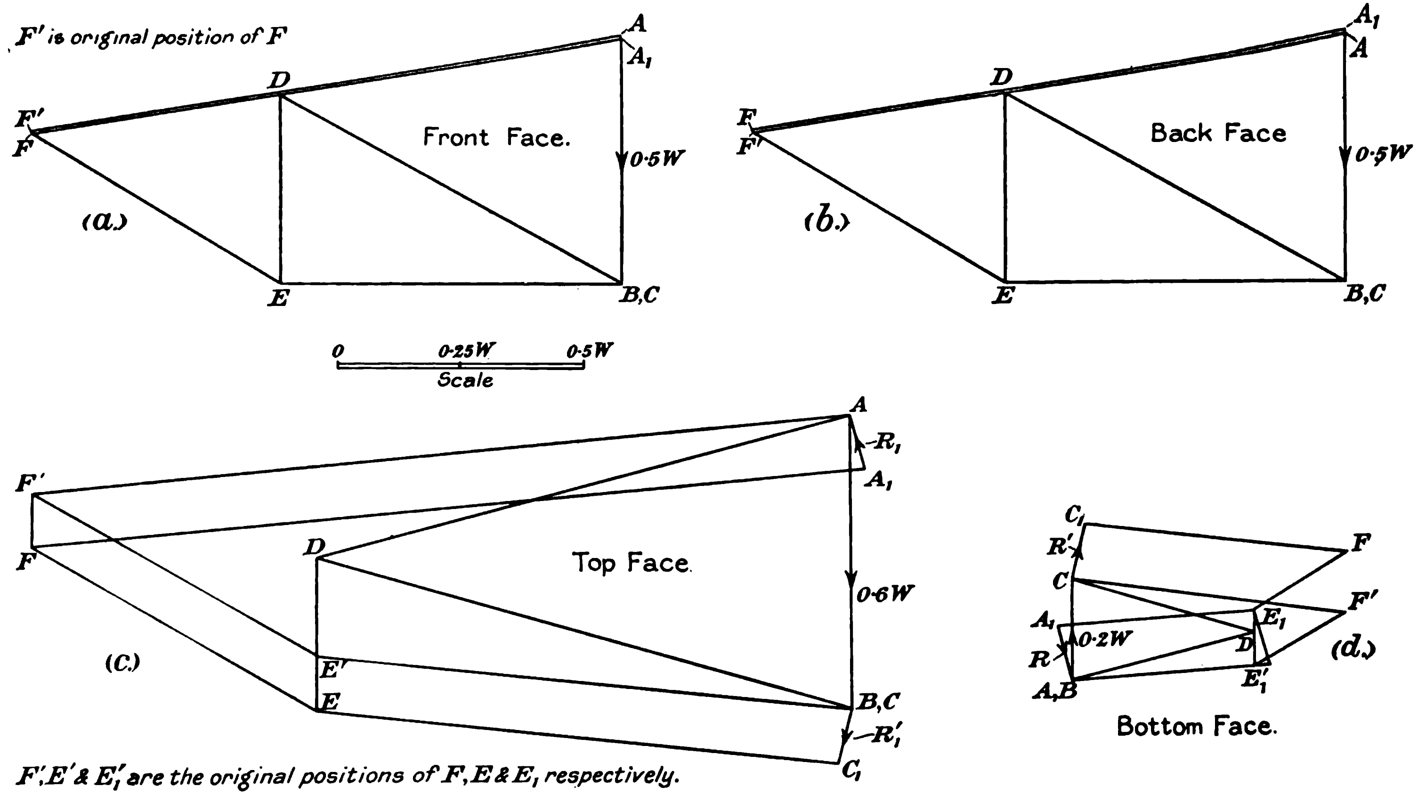


Fig. 103. Stresses in loaded rigid structures with redundancies.

straints. From fig. 103 *a* and 103 *d* the force in *bc* is given by the algebraic sum of A_1F, fig. 103 *a*, and A_1E_1, fig. 103 *d*. In the upper face the wire b_1c_1' is in tension, while in the lower face it is the wire cb'. In fig. 103 the difference between points such as F and F' represents the effect on the force distribution of the constraints. By the same process we may proceed step by step along the whole fuselage. It is evident that should any one of the bracing wires go out of action by appearing as a compressive element in the calculation, and therefore its neighbour enter, it will merely be necessary to repeat the calculation for that compartment, irrespective of previous steps, on the assumption that the neighbouring wire in the stress calculation is capable of taking up compression. This method can be carried through with rapidity and considerable accuracy. Fig. 103 and the table on p. 193 indicate that the effect of the constraints is comparatively small.

§ **29.** *Method of evaluating Forces in Redundant Members.* The difficulty of selecting the redundancies in any framework has already been discussed at some length, and before proceeding with an analysis of their effects this must be overcome. A redundancy, existing as it does in general to supply additional strength to the structure, depends for its operation upon the elastic properties of the material of the framework, in so far as these determine the deflections. The method that is usually adopted in treating problems of this nature is that known as the strain energy method. In this method the redundant members are first fixed upon, supposed removed from the structure leaving a rigid non-redundant framework, and replaced by externally applied forces of unknown magnitude, but equivalent to those exerted by the eliminated members. The force distribution in the framework can now be obtained by direct resolution in terms of the unknown applied forces and the strain energy function built up, viz: $\phi = \Sigma \frac{1}{2} \frac{P^2 l}{EA}$ where P is the force in a member of which E, l, and A are the elasticity, length and cross-sectional area respectively. The forces are then obtained by differentiating this function with respect to each redundant force and equating to zero. The assumption involved in this process is that the expression ϕ gives the strain energy of the structure, and attains a minimum when the externally applied force replacing the redundant member becomes

Forces in Terms of W.

Bay	Member		Plane Frame	Constrained Frame
1	Strut	aa_1	0	0
	Longeron	ab	−1·08	−1·08
	,,	a_1b_1	+1·13	+1·13
	,,	$a'b'$	−0·32	−0·32
	,,	$a_1'b_1'$	−1·13	−1·13
	Wire	ab_1	0·79	0·79
	,,	ab_1'	0·79	0·79
2	Strut	bb_1	−0·395	−0·395
	,,	$b'b_1'$	−0·395	−0·395
	,,	bb'	−0·07	+0·06
	,,	b_1b_1'	−0·2	−0·328
	Longeron	bc	−1·82	−1·80
	,,	b_1c_1	+1·78	+1·76
	,,	$b'c'$	−0·89	−0·87
	,,	$b_1'c_1'$	−1·01	−1·03
	Wire	bc_1	0·6	0·6
	,,	$b'c_1'$	0·6	0·6
	,,	$b_1'c_1$	0·69	0·69
	,,	bc'	0·23	0·23
3	Strut	cc_1	−0·318	−0·318
	,,	$c'c_1'$	−0·318	−0·318
	,,	cc'	−0·124	−0·07
	,,	c_1c_1'	−0·372	−0·415
	Longeron	cd	−2·31	−2·29
	,,	c_1d_1	+2·88	+2·85
	,,	$c'd'$	−0·95	−0·97
	,,	$c_1'd_1'$	−1·09	−1·12
	Wire	cd_1	0·43	0·43
	,,	$c'd_1$	0·43	0·43
	,,	$c_1'd_1$	0·72	0·72
	,,	cd'	0·24	0·24
4	Strut	dd_1	−0·267	−0·267
	,,	$d'd_1'$	−0·267	−0·267
	,,	dd'	−0·131	−0·131
	,,	d_1d_1'	−0·392	−0·392
	Longeron	de	−2·71	−2·69
	,,	d_1e_1	+3·81	+3·78
	,,	$d'e'$	−1·04	−1·02
	,,	$d_1'e_1'$	−1·21	−1·24
	Wire	de_1	0·33	0·33
	,,	$d'e_1'$	0·33	0·33
	,,	$d_1'e_1$	0·59	0·59
	,,	de'	0·197	0·197
5	Strut	ee_1	−0·23	−0·23
	,,	$e'e_1'$	−0·23	−0·23
	,,	ee'	−0·116	−0·116
	,,	e_1e_1'	−0·35	−0·35
	Longeron	ef	−3·03	−3·01
	,,	e_1f_1	+4·52	+4·49
	,,	$e'f'$	−1·08	−1·10
	,,	$e_1'f_1'$	−1·33	−1·36
	Wire	ef_1	0·27	0·27
	,,	$e'f_1'$	0·27	0·27
	,,	$e_1'f_1$	0·50	0·50
	,,	ef'	0·167	0·167

identical with the force in that member. This statement, however, depends for its truth on the fact that Hooke's law applies. Actually the equations derived by differentiation are simply statements of the equality of the deflections of the redundant members under the forces acting in them, and those of the rest of the frame under the applied forces, including the equivalent forces from the redundant members. On account of this, the actual derivation of the strain energy is a useless step, since the deflection equations can be written down directly without explicitly involving the assumption of the truth of Hooke's law. Thus, suppose the redundant members replaced by forces P, Q, etc., as before. Let F be the force in any non-redundant member, and suppose the latter replaced by it. Then from the geometry of the structure

$$F = aP + bQ + \ldots + \Sigma\, kW_1,$$

where P, Q, etc., are the forces in the redundant members supposed externally applied, and W_1, etc., are the real external loads. Since the deflections are supposed so small that the geometry is unchanged, this expression for F may be presumed unaltered after deflection.

Let δ_p, δ_q. etc., δ_w, be the deflections of the points of application of the forces P, Q, etc., W_1 due to an extension between the points of application of F (*i.e.*, of the member containing F), the work done under these deflections must be equal to zero. Hence

$$Fe = P\delta_p + Q\delta_q + \ldots + W\delta_w,$$

$$\therefore\ \delta_p = ae$$

$$\delta_q = be$$

etc.

The total deflection of P due to the extensions of all the members will be

$$\Sigma ae = -\text{ total deflection of the member under its own force}$$

$$= -\frac{Pl}{EA}.$$

It follows that the equation may be written down by equating minus the extension due to P of the member containing P to the sum of the products of the extensions of each member, when P is unity.

CHAPTER IX

CRITICAL BEHAVIOUR OF STRUCTURES

§ 1. In the previous chapter an illustration has been given, and cases may be multiplied, of a rigid structure of such geometrical form, that whenever any finite loading whatsoever is applied to it the forces brought into being in the members are so large that the structure must collapse. An inherent assumption, however, in this illustration is the fact that the elastic properties of the material are presumed not to come into play. Under such ideal conditions it may be stated that such a geometrical structure acts in a critical manner under any loading. If, however, the members be composed of an elastic material such as wood or metal, the critical geometry, while still operating to bring into play excessively large forces and deflections, will not necessarily involve the complete collapse of the structure on account of the counterbalancing effects of the elastic properties, but will become apparent by deflections excessively large in comparison with those of a non-critical structure. It is generally assumed in an analysis of the stresses in structures, as, for example, occurs in the beam theory, that the deflections which occur are always small in comparison with the lengths of the parts. By this means, justified on physical grounds, simplifications are introduced into the mathematical analysis which render it capable of treatment. It becomes at once evident therefore, that where critical conditions such as those indicated arise, these assumptions are at once invalid, but paradoxically enough, instead of rendering such cases incapable of treatment, the analysis indicates at once where they arise in virtue of the state of affairs resulting from the violation of the fundamental assumption of small deflections. In fact the treatment of critical cases resolves itself into the determination of the conditions under which the deflections, assumed in the analysis small, become apparently infinite in comparison.

§ 2. In general the occurrence of criticals will depend on two factors—the geometry of the structure and the elastic properties

of the material. Included in the geometry there must be the nature, magnitude and direction of the external loading. Everything in fact that may be represented by a line drawing of the problem. Two illustrations probably familiar to the student may here be brought to his notice. If a strut of uniform cross-section and homogeneously elastic be simply supported at the ends and placed under end thrust as in the case of the gap struts in the previous chapter, the end thrust, presumed for the moment eccentrically placed, may be increased up to a certain definite amount without the deflections becoming other than small. At this value the deviation from the straight becomes excessively large. Under these circumstances the strut is usually referred to as being critically loaded.

A second illustration will probably spring to the memory of the engineering student. If an ordinary shaft supported at the ends has its rotational speed increased from zero upwards, the central axis of the shaft remains undeflected or very approximately so until a certain rotational speed is reached in the neighbourhood of which the curvature of the shaft very rapidly increases and the latter is said to whirl. If the speed be rapidly increased above this point the deflections rapidly diminish and for a further range of speed the shaft remains straight until once more a higher speed is attained at which deflections again become excessively large and whirling is reproduced. The various rotational speeds at which this whirling occurs are referred to as critical speeds.

§ **3.** The whole theory of flexure of beams as usually treated is based on the assumption that the forces brought to play upon them are such as to give rise to only small deflections, and small curvatures. If anything, however, is striking about a critical, it is that the deflections are certainly not small but extremely large in comparison to the deflections obtained under normal conditions, and therefore at a first glance it would appear to be evident that the ordinary theory specially excluding this possibility, could give us no information with reference to the state of affairs which we seek to investigate. It is just the fact that the treatment of these cases is excluded from the ordinary theory that enables us by means of it to determine the existence of these criticals. The detailed nature of the behaviour of a strut, the exact curve it deflects to, is not the subject of our present

investigation, but merely the particular value of the end thrust at which a strut of any given variation in flexural rigidity, gives this extremely large deflection from the normal.

The real crux of the matter will therefore be settled when the criterion that must be applied to the ordinary theory of small deflection is fixed.

Without entering into any elaborate proof of the matter, however, it is clear that what distinguishes crippling from ordinary cases of flexure is the inordinately large deflections and therefore bending moments obtained. To all intents and purposes they are too large to allow of a legitimate application of the ordinary theory. The mathematical treatment based on this theory would express such deflections and moments as of infinite magnitude, and therefore if we wish to state the mathematical criterion for crippling in an investigation, using the ordinary theory, we must express it by saying that the deflections and bending moments are of infinite magnitude. The conditions under which this occurs will determine the conditions under which crippling occurs.

§ 4. As a simple illustration of the direct application of these considerations to the crippling of a strut of variable flexural

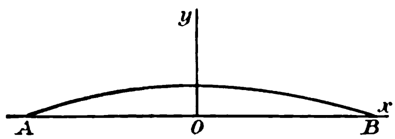


FIG. 104

rigidity, let AB be the strut of length l, of flexural rigidity EI, varying along the length according to any given law. The bending moment on the section at P of coordinate (xy) due to the constant end thrust F is Fy. The restoring moment at any point P is

$$\text{Flexural Rigidity} \times \text{Curvature}$$

$$= -EI\frac{d^2y}{dx^2},$$

where the deflection is assumed so small that the curvature may be written $\dfrac{d^2y}{dx^2}$ instead of $\dfrac{d^2y}{dx^2}\Big/\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{3/2}$,

$$EI\frac{d^2y}{dx^2} = -Fy,$$

$$\frac{d^2y}{dx^2} + \frac{F}{EI}y = 0.$$

To determine the criticals we require to solve this equation subject to the end conditions, say that at $x = 0$ and $x = l$ the deflections have definitely assigned values while at intermediate positions they become infinitely great. Before showing how this may be performed with perfect generality consider the following particular case.

Take the strut which is one of the series given by (fig. 105) a symmetrical section of length l of the ellipse. The principal reason for selecting this illustration, apart from its practical utility, is that it will serve as a check on the theory as later developed. By variations in the magnitude of the quantities a and b an infinite series of struts can be obtained, including very close approximations to those of most frequent occurrence in aeroplane practice.

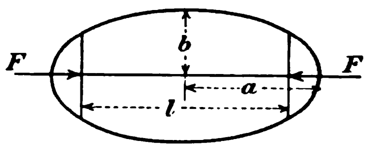


FIG. 105

The ordinates are then given by

$$\frac{y}{b} = \left(1 - \frac{x^2}{a^2}\right)^{1/2}.$$

Since the moments of inertia are proportional to the fourth power of the ordinates, this gives

$$\frac{I}{I_0} = \left(1 - \frac{x^2}{a^2}\right)^2,$$

where I_0 = moment of inertia at central section.

The equation whose solution is required subject to the criterion of crippling is

$$\frac{d^2y}{dx^2} + \frac{Fy}{EI_0\left(1 - \frac{x^2}{a^2}\right)^2} = 0.$$

The equation is completely integrable giving

$$y = (a^2 - x^2)^{1/2}\left[A\left(\frac{a+x}{a-x}\right)^\lambda + B\left(\frac{a-x}{a+x}\right)^\lambda\right],$$

where $$\lambda = \frac{i}{2}\left[\frac{Fa^2}{EI_0} - 1\right].$$

The deflection curve will thus be completely specified provided A and B be determined from the end conditions. It becomes at once apparent that unless a slight deflection be given to one or other of the ends ideally no flexure whatever will take place, for A and B will both be zero. Let us assume then that at

$$x = -l/2, \quad y = 0$$

and at

$$x = +l/2, \quad y = a.$$

Crippling will then occur when the deflection at any point becomes infinitely great in comparison with a.

Inserting the end conditions this gives

$$0 = \left(a^2 - \frac{l^2}{4}\right)^{1/2} \left[A\left(\frac{a - l/2}{a + l/2}\right)^{\lambda} + B\left(\frac{a + l/2}{a - l/2}\right)^{\lambda}\right],$$

$$a = \left(a^2 - \frac{l^2}{4}\right)^{1/2} \left[A\left(\frac{a + l/2}{a - l/2}\right)^{\lambda} + B\left(\frac{a - l/2}{a + l/2}\right)^{\lambda}\right],$$

providing two equations to solve for A and B.

Performing the algebraical solution it is easily seen that the expressions for both A and B will have as denominator

$$\left(\frac{a - l/2}{a + l/2}\right)^{2\lambda} - \left(\frac{a + l/2}{a - l/2}\right)^{2\lambda}.$$

If the deflections are to become infinite, A and B must become infinite, and therefore the condition for crippling takes the form

$$\left(\frac{a - l/2}{a + l/2}\right)^{2\lambda} - \left(\frac{a + l/2}{a - l/2}\right)^{2\lambda} = 0,$$

i.e.

$$\left(\frac{2a + l}{2a - l}\right)^{4\lambda} = 1.$$

It must be remembered that λ contains the imaginary quantity i. so that there are other solutions of this equation in λ besides $\lambda = 0$. In fact

$$e^{4\lambda \log\left(\frac{2a+l}{2a-l}\right)} = 1$$

and remembering that

$$e^{2n\pi i} = 1,$$

where n is any integer, it follows that

$$\lambda \log\left(\frac{2a + l}{2a - l}\right) = n\pi i$$

or

$$\left(\frac{Fa^2}{EI_0} - 1\right)^{1/2} \log\left(\frac{2a + l}{2a - l}\right) = n\pi,$$

giving
$$F = \frac{EI_0}{a^2}\left[1 + \frac{n^2\pi^2}{\left(\log\frac{2a+l}{2a-l}\right)^2}\right]$$

as the expression for the various crippling loads.

For convenience in calculation it is frequently better to write

$$r^2 = \frac{I_1}{I_0} = 1 - \frac{l^2}{4a^2}\,.$$

$$= \frac{\text{moment of inertia at end}}{\text{moment of inertia at centre}}\,,$$

so that
$$F = 4EI_0\left(\frac{1-r}{l^2}\right)\left[1 + \frac{n^2\pi^2}{\left(\log_e\frac{1+\sqrt{1-r}}{1-\sqrt{1-r}}\right)^2}\right].$$

The limit of this expression for $r = 1$ is easily shown to give

$$F = \frac{EI_0}{l^2}n^2\pi^2$$

which is Euler's value.

The case for $n = 0$ must be excluded for this would give

$$Fa^2 = EI_0,$$

and the solution obtained is illegitimate. The first crippling load is thus

$$F = \frac{4EI_0}{l^2}(1-r)\left[1 + \frac{\pi^2}{\left(\log_e\frac{1+\sqrt{1-r}}{1-\sqrt{1-r}}\right)^2}\right].$$

When r is made to differ slightly from unity so that the strut is nearly straight this gives

$$F = \frac{E}{l^2}(8{\cdot}58\,I_0 + 1{\cdot}29\,I_1).$$

When $r = 1/2$ so that the moment of inertia at the end is only $1/4$ that at the mid-section

$$F = \frac{4EI_0}{l^2}\cdot\frac{1}{2}\left[1 + \frac{\pi^2}{\left(\log_e\frac{1+\sqrt{1/2}}{1-\sqrt{1/2}}\right)^2}\right]$$

$$= \frac{EI_0}{l^2}\times 8{\cdot}22.$$

But for a uniform strut

$$F = \frac{EI_0}{l^2}\pi^2 = 9{\cdot}86\,\frac{EI_0}{l^2}.$$

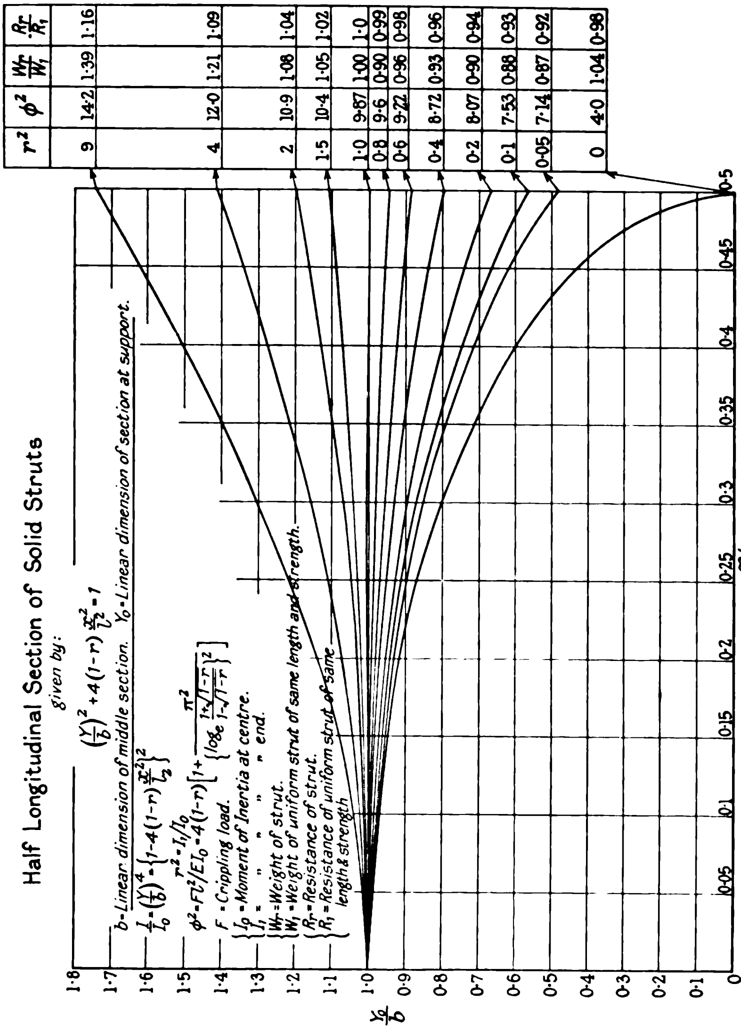


FIG. 106

Hence by tapering the strut to a moment of inertia at the tip equal to 1/4 that at the central section the strength has been reduced by 16 per cent. In fig. 106 the linear dimensions of the similar sections of the struts are plotted—ellipses and hyperboles—for

various values of r, and a comparison made in each case for the weights and aerodynamic resistances between the strut in question and a uniform one of equal strength and length. All cross-sections are calculated by taking them proportional to the projected area of the strut normal to the stream. It appears at once that in the neighbourhood of $r^2 = 0{\cdot}05$ both the relative weights and resistances are simultaneously a minimum. This therefore corresponds to the best strut of minimum weight if its sections are so selected as regards drag as to be most nearly streamline.

§ 5. Returning to the general problem of the crippling of a strut of any given law of variation in flexural rigidity, the equation to be solved is

$$\frac{d^2y}{dx^2} + \frac{F}{EI}y = 0,$$

where I is some given function of x.

In the drawing office the law of moment of inertia will not generally be specified by means of an analytic expression but rather in the more convenient form of a curve. The method of solution therefore most suitable for such conditions will evidently be one depending on graphical methods of integration.

Without entering into any proof of the matter the solution of the given equation may be written

$$y = A\left\{1 - F\int_0^x dx\int_0^x \frac{dx}{EI} + F^2\int_0^x dx\int_0^x \frac{dx}{EI}\int_0^x dx\int_0^x \frac{dx}{EI} - \dots\right\}$$
$$+ B\left\{x - F\int_0^x dx\int_0^x \frac{x\,dx}{EI} + F^2\int_0^x dx\int_0^x \frac{dx}{EI}\int_0^x dx\int_0^x \frac{x\,dx}{EI} - \dots\right\},$$

where A and B are arbitrary constants and the terms in the two infinite series are each two integrations higher than the preceding one according to a simple law.

We may write this

$$y = AU(x) + BV(x).$$

A and B are to be obtained from the end conditions, viz.

$$x = 0, \quad y = y_0,$$
$$x = l, \quad y = y_1,$$

where the origin of coordinates is now taken at the end of the strut. Hence

$$y_0 = AU(0) + BV(0),$$
$$y_1 = AU(l) + BV(l).$$

When these equations are solved for A and B, it is seen that they have the common denominator

$$U(0)\,V(l) - V(0)\,U(l).$$

As already explained, crippling will occur when the deflections become infinite in comparison with y_0 and y_1, and this demands that the above expression vanishes.

Hence $$U(0)\,V(l) - V(0)\,U(l) = 0.$$

But from the expressions for U and V it is clear

$$U(0) = 1,$$
$$V(0) = 0.$$

Hence the condition for crippling becomes

$$V(l) = 0,$$

i.e. $$l - F\int_0^l dx \int_0^x \frac{x\,dx}{EI} + F^2 \int_0^l dx \int_0^x \frac{dx}{EI} \int_0^x dx \int_0^x \frac{x\,dx}{EI} - \text{etc.} = 0.$$

This is an equation containing an infinite number of terms as an equation to solve for F the crippling force.

But the formula is not nearly so formidable as at first sight appears.

In the first place, each of the terms in this expression can be obtained with great rapidity by a simple process involving nothing more than the use of an ordinary planimeter, or the integration of a curve by the mean ordinate method. In the second place it can be proved that the series is so rapidly convergent that no more than the first few terms, no more in fact than the terms actually given above, need be treated and the equation may be considered as a quadratic in F, the lowest root being taken. A first check on the validity of this series may be made by evaluating the expression for the case of a uniform strut, where EI is constant. The series then becomes

$$l - \frac{F}{EI}\int_0^l dx \int_0^x x\,dx + \frac{F^2}{E^2I^2}\int_0^l dx \int_0^x dx \int_0^x dx \int_0^x x\,dx - \text{etc.} = 0$$

i.e. $$l - \frac{Fl^3}{EI}\cdot\frac{1}{3!} + \frac{F^2l^5}{E^2I^2}\cdot\frac{1}{5!} - \dots = 0,$$

or $$\phi - \frac{\phi^3}{3!} + \frac{\phi^5}{5!} - \dots = 0,$$

after multiplying by $\sqrt{(F/EI)}$, where $\phi = l\sqrt{F/EI}$

$$\sin \phi = 0,$$

i.e. $\phi = \pi$, $\phi = 0$ not being a solution.

For convenience in carrying through the graphical integration it is advisable to throw the series obtained into a non-dimensional form. This may be done by transforming it by the substitution

$$X = x/l,$$

and writing

$$\phi^2 = Fl^2/EI,$$

where $I = I_0 R$, I_0 being the moment of inertia at the end, whence we obtain

$$1 - \phi^2 \int_0^1 dX \int_0^x \frac{X\,dX}{R} + \phi^4 \int_0^1 dX \int_0^x \frac{dX}{R} \int_0^x dX \int_0^x \frac{X\,dX}{R} - \dots = 0.$$

The graphical process is then extremely simple. The curve for X/R is plotted for the range $X = 0$ to $X = 1$, the whole length of the strut, and the value of the integral obtained graphically by means of a planimeter or by mean ordinates is plotted. This determines $\frac{X\,dX}{R}$ which is once more integrated for the range 0 to 1, and plotted. The value of the ordinate at $X = 1$ determines the first coefficient, viz. $\int_0^1 dX \int_0^x \frac{X\,dX}{R}$.

Each ordinate of the last curve is then multiplied by $1/R$ and integrated twice as before, and the value of the final ordinate at $X = 1$ determines the coefficient of ϕ^4. In general it will not be found necessary to proceed to higher terms but if necessary this may be done with equal rapidity by multiplying by $1/R$ and integrating twice. After a little practice it will be found that this whole process apparently complicated can be performed in half an hour. The final operation obtained is then in the form

$$1 - a\phi^2 + b\phi^4 - \dots = 0,$$

which, when solved for ϕ^2 and selecting the lowest positive root, gives the value of Fl^2/EI_0 which corresponds to the crippling load.

The foregoing provides what is in practice a rapid yet effective method of estimating the crippling load for a column of any given shape. In aeronautical applications, however, struts are invariably symmetrical about a plane perpendicular to the axis and the work consequently may be considerably simplified.

The symmetry may be introduced quite early by using as the end conditions to determine the constants A and B in the expression for y the solution of the differential equation, that

$$\text{at } x = 0, \qquad y = 0,$$
$$\text{at } x = l/2, \quad \frac{dy}{dx} = 0.$$

If this be done the condition for crippling then takes the form

$$U(0)\,V'(l/2) - V(0)\,U'(l/2) = 0,$$

i.e.
$$1 - F\int_0^l \frac{x\,dx}{EI} + F^2\int_0^l \frac{dx}{EI}\int_0^x dx\int_0^x \frac{x\,dx}{EI} - \ldots = 0.$$

The integration now extends over half the length of the strut and each coefficient is one integration less than in the asymmetric case.

As before if we throw the series into the non-dimensional form by writing

$$X = 2x/l,$$
$$I = I_0 R,$$
$$\phi^2 = Fl^2/EI_0,$$

it reduces to

$$1 - \frac{\phi^2}{4}\int_0^1 \frac{X\,dX}{R} + \frac{\phi^4}{16}\int_0^1 \frac{dX}{R}\int_0^x dX\int_0^x \frac{X\,dX}{R}$$
$$- \frac{\phi^6}{64}\int_0^1 \frac{dX}{R}\int_0^x dX\int_0^x \frac{dX}{R}\int_0^x dX\int_0^x \frac{X\,dX}{R} + \ldots = 0.$$

As an illustration of the application of this method, consider the strut given by $r = 1/4$ in the system of elliptical struts already treated (*i.e.* $I_1/I_0 = 1/16$ in our previous set).

For the strut the following table may be drawn up showing the various stages in the calculations.

A little explanation may be necessary for this table. The first three columns are of course derived from the dimensions of the given strut, while column IV is obtained by direct integration of the graph of column III. The coefficient of ϕ^2 in the series is thus $1/4 \times \cdot 757$. For graphical convenience column V is introduced where the maximum ordinate in IV is replaced by unity, all the other members being divided by ·757. This means that in the later integration the factor ·757 must be introduced.

Column VII is likewise derived from column VI by multiplying each term by the corresponding value of $1/R$ and reducing the maximum ordinate once more to unity. This again introduces

another factor. The coefficients of the terms that thus enter into the series are always found at the head of the columns to the left of the double vertical line.

X	$1/R$	X/R	$A = \int \frac{X\,dX}{R}$	A (reduced)	$B = \int A\,dX$	$\frac{B}{R} = C$ (reduced)	$\int C\,dX$	D (reduced)	$\int D\,dX$	E (reduced)	$\int E\,dX$
1·0	1·000	1·0	·757	1·0	·450	1·0	·393	1·0	·310	1·0	·300
·9	1·015	·914	·662	·875	·360	·81	·306	·778	·227	·741	·217
·8	1·062	·850	·575	·76	·280	·661	·235	·598	·160	·548	·156
·7	1·150	·805	·492	·65	·212	·542	·176	·448	·110	·408	·110
·6	1·292	·775	·414	·548	·153	·44	·128	·326	·071	·295	·075
·5	1·515	·758	·337	·445	·105	·353	·088	·224	·045	·219	·050
·4	1·878	·751	·261	·345	·066	·276	·058	·148	·026	·157	·030
·3	2·500	·750	·187	·247	·036	·200	·032	·0814	·013	·105	·017
·2	3·700	·74	·111	·147	·016	·135	·015	·038	·006	·0715	·008
·1	6·50	·65	·042	·056	·004	·058	·004	·010	·002	·0420	·003
0	1·60	0	0	0	0	0	0	0	0	0	0
I	II	III	IV	V	VI	VII	VIII	IX	X	XI	XII

Hence

Coefficient of $\phi^2 = 1/4 \times \cdot 757 = \cdot 189.$

„ „ $\phi^4 = 1/16 \times \cdot 757 \times \cdot 45 \times \cdot 393 = \cdot 00837.$

„ „ $\phi^6 = 1/64 \times \cdot 757 \times \cdot 45 \times \cdot 393 \times \cdot 300 = \cdot 0001945.$

Equation to be solved is consequently

$$1 - \cdot 189\phi^2 + \cdot 00837\phi^4 - \cdot 00019\phi^6 + \ldots = 0.$$

Neglecting the term in ϕ^6 and solving as a quadratic this gives

$$\phi^2 = 8\cdot 6 \text{ approx.}$$

To determine a closer approximation to the root let

$$\phi^2 = 8\cdot 6 + \lambda.$$

Inserting this in the equation and neglecting higher powers of λ than the first we find

$$1 - \cdot 189\,(8\cdot 6 + \lambda) + \cdot 00837\,(8\cdot 6^2 + 17\cdot 2\lambda) - \cdot 00019\,(8\cdot 6^3 + 3 \times 8\cdot 6^2) + \ldots = 0,$$

i.e. $\lambda = -\,1\cdot 40.$

Hence $\phi^2 = 7\cdot 2$

giving as the crippling load

$$F = 7\cdot 2\,\frac{EI_0}{l^2}.$$

For comparison and as a check on the result we can evaluate ϕ^2 from the previous analysis of the elliptical set of struts when we find

$$\phi^2 = 7\cdot 25.$$

§ **6.** *Practical considerations in the Design of a Strut.* The foregoing analysis has presumed that failure can only ultimately take place due to buckling or instability of a particular nature and has ignored the possibility of other factors entering in which would tend to cause collapse other than of this nature. For example, if the yield point of the material were reached under an end thrust far below the value already calculated it is evident that Euler's value in case of a uniform strut might give no indication of where this rupture would be expected. The fundamental assumption in fact has been made that E remains a definite elastic constant right up to the crippling load, an assumption clearly invalid if the yield stress should be reached before that point.

It is because of the possibility of the occurrence of failure due to these two causes, crippling and yielding, that one cannot look with perfect assurance to the value estimated in the previous analysis as representing the actual collapsing load. For the case

of uniform struts of various cross-sections much experimental work has been carried through with a view to determining what deviations, if any, occur from Euler's value. Gordon, as a result of a series of tests on circular struts, deduced the following empirical formula,

$$F = \frac{f_c A}{1 + a \frac{l^2}{d^2}},$$

where

f_c is the crushing stress of the material,

l is the length,

d the diameter,

and A the area of cross-section of the strut.

a = a constant $= f_c/\pi^2 E$, approximately.

This indicates that for short struts crushing is the main factor in collapse, while for longer struts Euler's value approximately holds good. An extension by Rankin to struts of any cross-section gives the formula

$$F = \frac{f_c A}{1 + a\left(\frac{l}{K}\right)^2},$$

where K is the radius of gyration and

$$a = f_c/\pi^2 E.$$

This is an attempt to take into account stresses due to direct thrust and those due to bending, but unfortunately such a formula applied to struts of any cross-section, as Rankin attempts to do, must necessarily be inadequate for the stresses due to bending will evidently depend upon the depth of section, a factor finding no place in his formula. Much experimental work has since then been carried through to investigate the deviations and causes of deviation from Euler for a uniform strut. It has become increasingly clear that such differences as do occur are in almost all circumstances completely explicable in terms of such unavoidable factors as lack of homogeneity of material, unavoidable eccentricities in loading, nature of supports and junctions. Such eccentricities of an indefinite nature can only be allowed for by an adequate factor of safety. Agreement with the Euler's value for a uniform strut has been found to depend almost entirely upon the extent to which such inherent factors just referred to are absent.

As an illustration of the nature of the results that have been obtained, the following table, based on experiments on aeroplane struts of half scale, may be taken as typical.

Material	Equivalent weight of full-sized strut (lbs.)	Equivalent load of full-sized strut (lbs.)	$\frac{\text{load}}{\text{weight}}$ spruce $=1$	Equivalent load by Euler	$\frac{\text{Actual load} \times 100}{\text{Load by Euler}}$
Silver Spruce	5·38 5·55	2·116 2·304	1·00	2·380 2·400	93
Victorian Blackwood (Light colour)	8·07 7·56	2·876 2·796	0·90	2·700 2·520	109
Victorian Blackwood (Dark colour)	8·52 9·24	3·732 3·880	1·06	4·090 3·850	97
Wollybutt	13·07 13·09	3·932 4·108	0·76	4·090 5·050	97
Ironbark	14·21 14·39	4·604 4·788	0·81	5·070 4·820	95
Blackbutt	13·06 13·14	4·508 4·604	0·86	4·660 4·740	97

§ **7.** Before concluding this section it is important to note that the general analysis of the simple strut has centred round the evaluation of the non-dimensional quantity $\phi^2 = Fl^2/EI$. In the case of a uniform strut ϕ has been shown to equal π while the value

of the quantity for a non-uniform strut depends upon the shape of that member. In a later investigation it will be found that the conception of the existence of such a non-dimensional constant is not limited to this simple case but is of general significance in the whole theory of structures and of particular import in the theory of aeroplane structures.

§ 8. Consider the case of a structure of the type of an aeroplane framework for example, where it is supposed that the assumptions made in the ordinary beam theory apply to every part. Let the length of one part, say a bay, be given by l, the area of a particular section by A, moment of inertia I, and the elasticity and density of the material of which it is composed E and ρ respectively. Given these quantities for this one portion, it will be assumed that the shape and geometry of the structure involve an exact specification of how to derive the corresponding quantities for the remaining portions. Let the external load be F, applied in some given manner specified by the geometry, and suppose this is sufficient to produce a stress just greater than the yield stress f in the material of the weakest member. F can only depend apart from the geometry and manner of application upon the quantities enumerated above, defining the properties of the material. These are as follows:

Physical quantity	Symbol	Dimensions
Young's Modulus	E	M/LT^2
Area	A	L^2
Moment of Inertia	I	L^4
Length	l	L
Density	ρ	M/L^3
Yielding stress	f	M/LT^2
Gravity	g	L/T^2
Load	F	ML/T^2

Since F, the load which will cause collapse by rupture of the material, must involve all these quantities grouped together in such a manner as to make the dimensions of the product identical with those of F, we may write

$$F \propto E^x A^y I^z l^p \rho^q f^r g^s.$$

The dimensions of this are clearly

$$M^{x+q+r} L^{-x+2y+4z+p-3q-r+s} T^{-2x-2r-2s} = \frac{ML}{T^2}.$$

Equating powers of M, L and T,

$$x + q + r = 1,$$
$$-x + 2y + 4z + p - 3q - r + s = 1,$$
$$x + r + s = 1,$$
$$\therefore F \propto El^2 \left(\frac{f}{E}\right)^r \left(\frac{A}{l^2}\right)^y \left(\frac{gl\rho}{fA}\right)^s \left(\frac{I}{l^4}\right)^z.$$

It will be more convenient to write this in the form

$$F \propto \frac{EI}{l^2} \left(\frac{f}{E}\right)^{r+s} \left(\frac{A}{l^2}\right)^{y+s} \left(\frac{W}{fA}\right)^s \left(\frac{I}{l^4}\right)^{z-1},$$

where instead of $gl^3\rho$ its equivalent W, the weight of the portion of the structure considered, is taken.

It is evident that every term in the expression for F must take the form given above, but so far no restriction has been imposed on the values of the quantities r, s, y and z. Under these conditions F could be represented as the sum of a series of terms in which f/E, A/l^2, W/fA, I/l^4 could appear to any power whatsoever but in each such term EI/l^2 would be a factor. This is equivalent to the statement that

$$F = \frac{EI}{l^2} \psi \left(\frac{f}{E}, \frac{A}{l^2}, \frac{W}{fA}, \frac{I}{l^4}\right),$$

where ψ for the moment is an arbitrary function. Writing

$$\phi^2 = Fl^2/EI \qquad \text{.........................(1)}$$

so that this quantity may be considered as a non-dimensional critical loading coefficient, it follows finally that

$$\phi^2 = \psi \left(\frac{f}{E}, \frac{A}{l^2}, \frac{W}{fA}, \frac{I}{l^4}\right) \qquad \text{..................(2).}$$

If it be desired to introduce in addition the depth d of the section, then this will merely give rise to the extra term d/l in the function.

It is clear from the definition of ϕ, that for a given structure where l, E and I are known ϕ is uniquely determined when F, the breaking load, is found, so that the discussion of collapse by rupture may equally well be centred round ϕ. Let it be clearly understood that the exact form of the function ψ in (2) depends only on the geometry of the structure, including the manner of loading and the law of distribution of material.

§ **9.** *Similarity of Structures.* **Out of the general class of framework embraced in this discussion so far, let there be selected a series of which all members are identical with respect to external shape, differing only in magnitude. This implies that A the area of cross-section of any selected part is proportional to l^2 and I to l^4. For this series, therefore, A/l^2 and I/l^4 are constants depending only on the geometry, and accordingly expression (2) now takes the simplified form**

$$\phi^2 = \psi_1\left(\frac{f}{E}, \frac{W}{fA}\right) \quad \text{.....................(3).}$$

It is not difficult to give an interpretation to the two non-dimensional quantities f/E and W/fA. The former being the ratio of the yielding stress to Young's modulus may be taken as the strain at the yield point, on the assumption that Hooke's Law applies rigorously over the whole range. In the same way W/fA is the ratio of the weight of a particular member, say the weakest, to the maximum tensile or compressive force, as the case may be, which that member is capable of withstanding without yielding assuming that the section A is taken at the position of collapse.

As far as a discussion of the breaking load coefficient ϕ^2 is concerned, it is evident that if a series of geometrically similar structures be selected, two distinct variables, and two only, functions of the material affect the question. These are the strain at the yield point for the weakest member, and the ratio of the weight of that member to the maximum tensile or compressive force it can withstand at the point of yield. It is equally clear, however, that the so-called geometrical restrictions determining the distribution of the constants of the materials in a composite structure are not absolutely vital to the above discussion, so that violations of the laws determining the selection of the material from member to member would not necessarily vitiate the conclusions arrived at here. For example, one would not be rigorously entitled to vary I, A and E in certain members, since such a change would clearly upset the geometrical similarity of shape and the distribution of force with the possible result that the previously weakest member might now not remain such. No such troubles, however, could arise if the selection of the materials was made without changing these factors, but allowing a variation in f the yielding stress so long as the previously weakest member always

remains so in the series. This is equivalent to an increased latitude in the scope of selection of materials.

Generally in experimenting with a model of this nature it is convenient, of course, to have corresponding parts in the model and the full-scale structure composed of identical materials. If, for example, the frame of an aeroplane were under discussion, wires in the model would be composed of the same metal as those in the original, corresponding struts, spars, ribs, etc., of the same wood.

Since the quantities f, E, l, ρ, etc., in the expressions for ϕ^2 all refer ultimately to the same part of the model, to ensure geometrical similarity in the widest sense it follows that the assumption that f/E is a constant is one which is obvious experimentally. Under these circumstances also W/fA, as can easily be verified, becomes proportional to the linear dimension l. The breaking load coefficient ϕ^2, therefore, when the weight of the structure contributes towards rupture, is purely a function of l,

$$\phi^2 = \phi_1 (l),$$

and the form of the function is determined only by the shape of the various parts, the geometry of the loading and the so-called geometry of the material.

This expression for ϕ^2 may likewise be supposed solved for l in terms of ϕ. For every external load coefficient this equation then determines the size of the structure corresponding to failure. When there is no external load and failure is due to weight alone $\phi = 0$ and the result previously anticipated is seen to hold.

§ **10.** *Structures of Negligible Weight.* When the material is not sufficiently heavy to involve the weight of the structure appreciably as a factor in causing collapse, the term W/fA may be omitted in equation (3) and the expression for ϕ^2 takes the relatively simple form

$$\phi^2 = \phi_2 \left(\frac{f}{E}\right).$$

Let us imagine that the weakest member is replaced by another of the same geometrical shape and size but with a different value of f/E, though not so different as to prevent its remaining the weakest member, then a test on each one of these models will give a particular value of ϕ^2, breaking load coefficient, and these when

plotted against f/E will give a "characteristic curve" for the structure of the given geometry. For geometrically similar structures of which the corresponding parts are made of the same material, so that f/E is constant, the critical load coefficient becomes an absolute constant for the series depending for its value purely on the shape. It follows that a simple test to destruction on a model will suffice under these conditions to determine the breaking load coefficient ϕ^2, and therefore the breaking load F for any other member of the series.

§ **11.** *Failure Due to Instability.* It is assumed in this section that the failure is not necessarily accompanied by collapse or rupture in the material but is due merely to permanent deformation of the configuration of the structure. It follows that in the expression for the critical load, the yield stress does not enter although all the other terms A, l, E, I, ρ, g may do so. Using ϕ^2 as a critical load coefficient defined by $Fl^2/EI = \phi^2$ then

$$\phi^2 = \psi_1\left(\frac{A}{l^2}, \frac{W}{EA}, \frac{I}{l^4}\right) \qquad \text{(4)},$$

obtained by the same method as in the previous section.

On the understanding that the structures in the series are all geometrically similar as regards external shape without reference to material composing them, then A/l^2 and I/l^4 are constants for the series and

$$\phi^2 = \psi_2\left(\frac{W}{EA}\right) \qquad \text{(5)},$$

where W/EA is the ratio of the weight of a member to the tensile or compressive force required to produce unit strain at some particular section of that member.

If corresponding members of the series are made of the same material then W/EA is easily seen to be proportional to l, from structure to structure, and as before the critical loading coefficient

$$\phi^2 = \psi_3(l) \qquad \text{(6)}$$

depends on the size only.

When the weight of the structure is negligible as far as its effect in contributing to failure is concerned, the whole instability arising from the external loading, then ϕ^2 is a constant for a series of structures of identical form irrespective of the material of which it is composed, and a test to destruction on one model suffices for the series.

It has been found that the critical loading for a uniform prismatic strut of negligible weight under end thrusts and pin-jointed at the ends is given by $F = \pi^2 \frac{EI}{l^2}$, where $l =$ length of strut, I the least moment of inertia, and $E =$ Young's modulus, so that ϕ for this structure is π. For a strut under the same conditions but with fixed ends $\phi = 2\pi$.

In certain cases of simple shapes of structures, it is obvious that the function ϕ^2 may be derived by calculation as for example in the case of the prismatic beam supported either on simple or clamped supports, but for more complicated problems where the calculation is too abstruse a number of points on the characteristic curve may be derived by a series of tests to destruction on models.

§ **12.** In one other type of structure at least other than that of a simple strut the crippling coefficient ϕ can be evaluated. The case referred to, of extreme importance in aeronautical stress calculations, is that which corresponds to the spars of a wing or the longerons of a fuselage. For purposes of calculation these members may be supposed, with sufficient accuracy for the present, to consist of a strut held by simple supports at a number of points along its length loaded laterally in any given manner and under the influence of end thrust varying from bay to bay in a manner determined by the general configuration and loading of the rest of the structure and inclinations and positions of attachment of wires to each bay. The actual values of these end thrusts must of course be calculated according to the principles enunciated in the previous chapter.

§ **13.** *Case of a longeron of n Bays.* Let there be $n + 1$ supports, figured 0, 1, 2, ..., n, and spaced as in the diagram, so that the rth bay appears to the left of the support numbered r. Let the bending moments at these supports be M_0, M_1, ..., M_n, the deflections as already defined in the previous chapter δ_1, δ_2, ..., δ_{n-1}, and suppose the lateral loading considered uniform along each bay, l_1, l_2, ..., l_n, is w_1, w_2, ..., w_n. E_1I_1, E_2I_2, ..., E_nI_n are as before the flexural rigidities of the various bays and F_1, F_2, ..., F_n the thrusts in these members.

Applying the generalised form of the equation of three moments,

derived in the previous chapter, to the three supports of every pair of contiguous bays, we obtain the following system of equations:

Central support of trio

1 $$M_0A_1 + M_1(B_1 + B_2) + M_2A_2 = \delta_1(1/l_1 + 1/l_2) - (K_1 + K_2) \quad ...(7_1),$$

2 $$M_1A_2 + M_2(B_2 + B_3) + M_3A_3 = \delta_2(1/l_2 + 1/l_3) - (K_2 + K_3) \quad ...(7_2),$$

....

r $$M_{r-1}A_r + M_r(B_r + B_{r+1}) + M_{r+1}A_{r+1} = \delta_r(1/l_r + 1/l_{r+1}) - (K_r + K_{r+1}) \quad ...(7_r),$$

....

$n-1$ $$M_{n-2}A_{n-1} + M_{n-1}(B_{n-1} + B_n) + M_nA_n = \delta_{n-1}(1/l_{n-1} + 1/l_n) - (K_{n-1} + K_n) \quad ...(7_{n-1}),$$

where A_r, B_r and K_r have the significance already attached to them in the previous chapter.

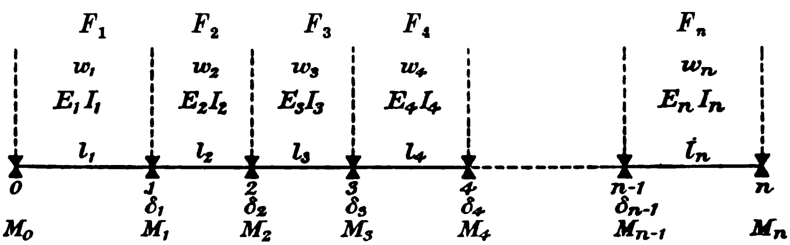


FIG. 107

These constitute $n - 1$ equations from which, if the bending moments at any two supports, usually 0 and n, be known, the remaining bending moments in general are uniquely determined.

§ **14.** *Case where one or more of the Bays are of Euler's Critical Length.* Suppose

$$\lambda_r l_r = \pi,$$

i.e. $$F = \pi^2 E_r I_r / l_r^2.$$

Such a bay, unassociated with the remainder of the structure, would collapse under the load. It is proposed to enquire whether or not the presence of the remainder of the structure can exert

a strengthening effect. The expression for the shape of the rth bay is given in the form

$$y = C_r \cos \lambda_r x + D_r \sin \lambda_r x - \frac{w_r}{2F_r}\left\{(l_r - x)^2 - \frac{2}{\lambda_r^3}\right\}$$
$$- \{M_{r-1} - f_{r-1}(l_r - x)\}/F_r \quad \text{......(8)},$$

where f_{r-1} satisfies

$$f_{r-1} = (M_{r-1} - M_r)/l_r + F_r \delta_r / l_r + \tfrac{1}{2} w_r l_r \quad \text{......(9)}.$$

Remembering that

$$x = 0, \quad y = \delta_r,$$
$$x = l_r, \quad y = 0,$$

and

$$\lambda_r l_r = \pi,$$

and inserting these in (8) modified by (9), we find

$$0 = C_r - M_r/F_r + \frac{w_r}{F_r \lambda_r^2},$$

$$0 = -C_r - M_{r-1}/F_r + \frac{w_r}{F_r \lambda_r^2}.$$

Therefore

$$C_r = -M_{r-1}/F_r + \frac{w_r}{F_r \lambda_r^2},$$

and

$$M_r + M_{r-1} = 2w_r/\lambda_r^2 = 2w_r l_r^2/\pi^2 \quad \text{............(10)}.$$

For no lateral load (10) takes the simplified form

$$M_r + M_{r-1} = 0 \quad \text{....................(11)}.$$

The constant D_r can at once be determined in terms of the M's by the statement that the gradient at the $(r-1)$th support for the rth bay is equal to the gradient at the same point for the $(r-1)$th bay, the expression for the latter involving no indeterminateness if it is not itself of Euler's length. If, however, the $(r-1)$th bay is also of Euler's length, then that bay must be treated in the same way as the rth above, with reference to the $(r-2)$th. It follows that all the coefficients are determinate, provided that one bay is not of Euler's length. Corresponding with each Euler's bay there will, of course, exist a condition of the form given in (10), and under these circumstances the structure will not fail provided it can be shown that none of the bending moments become infinite for any value of the longitudinal load F up to that here considered. This will come out from a discussion of the equations for bending moments.

It will be necessary to investigate the limiting forms which the coefficients A_r and B_r assume in this case. It is clear, of course, that both these quantities become in the limit infinitely great, but

$$\lim_{\lambda_r l_r \to \pi} (A_r - B_r) = \lim_{\lambda_r l_r \to \pi} \left\{ \frac{l_r}{E_r I_r} \cdot \frac{1}{\lambda_r^2 l_r^2} \left(\frac{\lambda_r l_r}{\sin \lambda_r l_r} + \frac{\lambda_r l_r}{\tan \lambda_r l_r} - 2 \right) \right\}$$

$$= \lim_{\lambda_r l_r \to \pi} \left\{ \frac{l_r}{E_r I_r} \left(\frac{1 + \cos \lambda_r l_r}{\sin \lambda_r l_r} - \frac{2}{\lambda_r l_r} \right) \right\} \frac{1}{\lambda_r l_r}$$

$$= \lim_{\lambda_r l_r \to \pi} \left\{ \frac{l_r}{E_r I_r} \cdot \frac{1}{\lambda_r l_r} \left(\frac{1}{\tan \frac{1}{2} \lambda_r l_r} - \frac{2}{\lambda_r l_r} \right) \right\}$$

$$= -\frac{2 l_r}{\pi^2 E_r I_r} \quad \text{.................................(12).}$$

Equations (7_1) to (7_{n-1}) determine the bending moments, but (7_{r-1}) and (7_r) require special treatment on account of the limiting forms of A_r, B_r, and K_r. Setting out these two equations we obtain

$$M_{r-2} A_{r-1} + M_{r-1} B_{r-1} + (M_{r-1} B_r + M_r A_r + K_r)$$
$$= \delta_{r-1} (1/l_{r-1} + 1/l_r) - K_{r-1} \text{...(13),}$$

$$(M_{r-1} A_r + M_r B_r + K_r) + M_r B_{r+1} + M_{r+1} A_{r+1}$$
$$= \delta_r (1/l_r + 1/l_{r+1}) - K_{r+1} \quad \text{...(14).}$$

The groups of terms enclosed in the brackets to the left-hand side of these equations clearly require modification. The following relations have been found to hold for uniform loading:

$$\lim_{\lambda_r l_r \to \pi} (M_{r-1} + M_r) = \frac{2 w_r}{\lambda_r^2} \quad \text{.....................(15),}$$

and

$$\lim_{\lambda_r l_r \to \pi} (A_r - B_r) = -\frac{2 l_r}{\pi^2 E_r I_r} \quad \text{...............(16).}$$

Subtracting (13) from (14) we find

$$(M_{r-2} A_{r-1} + M_{r-1} B_{r-1}) - (M_r B_{r+1} + M_{r+1} A_{r+1})$$
$$+ (M_{r-1} - M_r)(B_r - A_r)$$
$$= \delta_{r-1} (1/l_{r-1} + 1/l_r) - \delta_r (1/l_r + 1/l_{r+1}) - K_{r-1} + K_{r+1}.$$

The only term which adopts an indeterminate form is

$$\lim_{\lambda_r l_r \to \pi} (M_{r-1} - M_r)(B_r - A_r) \to \frac{2 l_r}{\pi^2 E_r I_r} (M_{r-1} - M_r).$$

Equations (7_{r-1}) and (7_r) must now be replaced by

$$M_{r-1} + M_r = \frac{2 w_r l_r^2}{\pi^2} \quad \text{..................(17),}$$

and $$M_{r-2}A_{r-1} + M_{r-1}\left(B_{r-1} + \frac{2l_r}{\pi^2 E_r I_r}\right) - M_r\left(B_{r+1} + \frac{2l_r}{\pi^2 E_r I_r}\right) - M_{r+1}A_{r+1}$$
$$= \delta_{r-1}(1/l_{r-1} + 1/l_r) - \delta_r(1/l_r + 1/l_{r+1}) - K_{r-1} + K_{r+1} \quad (18).$$

The two equations (7_{r-1}) and (7_r), containing coefficients which become infinite or indeterminate, can now, therefore, be replaced by (17) and (18), in which all the bending moments occurring in the previous two are still present, but the coefficients are all finite and known. It follows that in general the system of equations (7_1), (7_2), ..., (7_{r-2}), (17), (18), ..., (7_{n-1}) will determine finite bending moments for all the supports, and, consequently, also for every point in the beam. Only under special conditions about to be treated will failure occur due to buckling.

Summing up, then, a beam supported at any number of simple supports will not fail, in general, through the bending moments becoming excessive, even if some of the bays are of Euler's critical length, provided at least one bay is not of this length. Whether or not this will correspond with a stable loading of the structure will depend on whether or not the actual longitudinal load, if any, that would produce infinite bending moments is greater or less than Euler's load for the bays in question. It appears then that the presence of bays of Euler's length is not in itself either a necessary or sufficient criterion for instability. An accurate investigation of the conditions under which failure will take place will be given and a comparison between the crippling load thus determined and Euler's load will decide whether instability exists at the latter position.

§ **15.** *Cases where one of the Bays is twice Euler's Length.* Proceeding on the lines of the previous analysis, it can easily be shown that at least one of the bending moments becomes infinite and the structure fails.

§ **16.** *Conditions of crippling of a Supported System.* The criterion of crippling which will be utilised in the present discussion is, as already explained, the production of infinite bending moments. Associated with this will be the assumption already known to be valid in the case of a single strut, that for values of the longitudinal force greater than the least that will produce

these infinite bending moments, the geometry of the structure will not be maintained, the failure, in this sense, thus corresponding with instability. It remains, therefore, to write down the mathematical expression which must be satisfied when the bending moments are infinite and to determine the least root, regarding it as an equation in ϕ, the crippling coefficient for the weakest bay.

§ 17. *Condition of crippling for the Case of n Bays,* i.e., $n+1$ *Supports* (fig. 107). The equations $(7_1), \ldots, (7_{n-1})$ are sufficient in general to determine the bending moments at the supports. These equations being linear, each of the bending moments may be expressed as the ratio between two determinants whose constituents are functions of the lengths of the bays, deflections, and the lateral and longitudinal loading. The bending moments will become infinite when, and only when, the denominator vanishes. This condition, assuming the bending moments at the end supports zero, expressed in determinantal form is:

$$0 = \begin{vmatrix} B_1+B_2, & A_2, & 0, & 0, \ldots\ldots & \ldots\ldots\ldots\ldots & 0, & 0 \\ A_2, & B_2+B_3, & A_3, & 0, \ldots\ldots & \ldots\ldots\ldots\ldots & 0, & 0 \\ 0, & A_3, & B_3+B_4, & A_4, \ldots\ldots & \ldots\ldots\ldots\ldots & 0, & 0 \\ 0, & 0, & A_4, & B_4+B_5, \ldots\ldots & \ldots\ldots\ldots\ldots & 0, & 0 \\ 0, & 0, & 0, & A_5, \ldots\ldots & \ldots\ldots\ldots\ldots & 0, & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & A_{n-3}, & 0, & 0 \\ 0, & 0, & 0, \ldots\ldots\ldots & A_{n-3}, & B_{n-3}+B_{n-2}, & A_{n-2}, & 0 \\ 0, & 0, & 0, \ldots\ldots\ldots & 0, & A_{n-2}, & B_{n-2}+B_{n-1}, & 0 \\ 0, & 0, & 0, \ldots\ldots\ldots & 0, & 0, & A_{n-1}, & B_{n-1}+B_n \end{vmatrix} \quad \ldots\ldots(19),$$

since $$M_1 = 0, \quad M_{n+1} = 0,$$

$$A_n = \frac{l_n}{E_n I_n} \cdot \frac{1}{\phi_n^2}\left(\frac{\phi_n}{\sin\phi_n} - 1\right), \quad B_n = \frac{l_n}{E_n I_n} \cdot \frac{1}{\phi_n^2}\left(1 - \frac{\phi_n}{\tan\phi_n}\right) \quad \ldots\ldots(20),$$

where $$\phi_n = \lambda_n l_n .$$

§ 18. For any particular case this symmetrical determinant can be very easily evaluated by the following scheme.

Writing the value of this determinant for the case bays of Nos. 1, 2, etc., as Δ_1, Δ_2, etc., we obtain:

		No. of bays
1	$= \Delta_1$	1
$(B_1 + B_2)\,\Delta_1$	$= \Delta_2$	2
$(B_2 + B_3)\,\Delta_2 - A_2{}^2\,\Delta_1$	$= \Delta_3$	3
$(B_3 + B_4)\,\Delta_3 - A_3{}^2\,\Delta_2$	$= \Delta_4$	4
$(B_4 + B_5)\,\Delta_4 - A_4{}^2\,\Delta_3$	$= \Delta_5$	5
....		
$(B_{n-1} + B_n)\,\Delta_{n-1} - A_{n-1}{}^2\,\Delta_{n-2} = \Delta_n$	......	n ...(21).

When the value of ϕ for any bay reaches π, A and B become infinite and accordingly the determinant in general also becomes infinite. To avoid this it would be convenient to divide throughout by the product of the A's, so that instead of taking $\Delta_n = 0$ as the determinantal equation the condition

$$D_n = \frac{1}{A_1 . A_2 . A_3 \ldots A_n} \Delta_n$$

may be used.

The determinantal condition (19), being an expression containing all the ϕ's, can then be easily expressed as a function of the ϕ for any particular bay. When the point of attachment of the wires is not on the neutral axis, a bending moment of known magnitude is introduced at the corresponding support, and the appropriate modification must then be inserted in the equations for three moments.

§ **19.** *Clamped Supports.* A clamped support may be supposed equivalent to two simple supports infinitely close together.

§ **20.** The nature of the crippling dealt with in the previous sections must be clearly borne in mind. The discussion has dealt specifically with collapse due to instability and has not contemplated failure of the structure due to the yield stress being reached. The modulus of elasticity in fact has been presumed to retain its constant value right up to its point of failure. It is evident, on the other hand, that if the lateral loading were increased sufficiently, stresses could be set up in the members sufficiently great to destroy the structure. This fact at once shows the necessity for discriminating between two classes of cases that arise in the aeroplane structure. In the case of the longerons of the

fuselage and in the case of the wing spar for bending in the plane of the wing, the lateral loading is so small that serious consequences due to the latter could not arise and the present form of analysis will correspond with considerable accuracy to what may be presumed to occur. In the case of bending of the wing spar in the normal plane on the other hand, conditions are different. The lateral loading in this case as already described in the previous chapter attains such proportions that failure, if it occurs, will take place due to the elastic limit being exceeded considerably before the region of the crippling load is reached. Ideally, from the fact that the lateral load does not enter into the constituents of the determinant of failure, the value of the crippling load is not affected by the magnitude of the lateral load.

§ **21.** *Failure of wing spars due to crippling in the plane of the wing.*

If abc and $a'b'c'$, etc., be the front and rear spar of a wing presumed connected by ribs sufficiently large in number to make the spars deflect in an identical manner, but yet so flexible as to add practically nothing to the flexural rigidity of the system, then the whole may be regarded as one strut or spar of total moment of inertia equal to the sum of the moments of inertia of both parts, and the thrust in any bay equal to the algebraic sum of the thrusts in corresponding constituents. The analysis then proceeds as already outlined for the longeron. In general the lateral load may be neglected with the exception of the case of a terminal dive of the aeroplane when the lateral load attains a considerable magnitude in this plane.

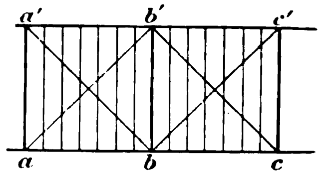


FIG. 108

PART IV

CHAPTER X

THE AIRSCREW

§ **1.** To maintain an aeroplane in flight, and thus produce the lifting force necessary to support the weight, the drag on the machine must be overcome by some horizontal propulsion. A propeller fulfils this function by discharging the air backwards from it and thereby receiving a forward momentum which it communicates to the remainder of the machine. In steady horizontal motion the thrust developed by this means will exactly balance the drag. It is the object of this chapter to investigate the precise manner in which this thrust originates, the interaction of air and propeller and the function which each portion of the blade fulfils. On these considerations as a basis, it should then be possible to proceed systematically towards the design of an efficient airscrew.

§ **2.** In earlier chapters it was seen how the motion of a body of aerofoil section in normal flying attitude gives rise to two component forces, a drag and a lift. If an aerofoil of small aspect ratio be supposed to move in a circular path and in approximately normal flying attitude about an axis parallel to the lift, the two component forces referred to will correspond respectively to a force parallel to the axis of rotation and one tangential to the circular path. As long as this circular motion is maintained against the dragging forces on the section, so long will there be developed a continuous thrust (lift) parallel to the axis of rotation. If now a series of such small aerofoils, not necessarily all of identical cross-sectional shape, be placed in normal flying attitude along a radius of the circle of rotation, and if these be revolved about the same axis as before, the lifting forces on the sections would be

directed along parallel lines. Generally speaking this may be regarded as a first approximation to a propeller. If continuously rotated it will produce a continuous thrust perpendicular to the disc of rotation, and a resisting torque in the plane of rotation.

§ 3. Fig. 109 *a* gives the plan form in the plane of rotation of a propeller blade showing the leading and trailing edges, the tip, and the boss. Fig. 109 *b* shows the developments of the sections made with the blade by a series of cylinders whose axis coincides with the axis of the propeller. The similarity of the sections, especially towards the tip, to that of an aerofoil is evident.

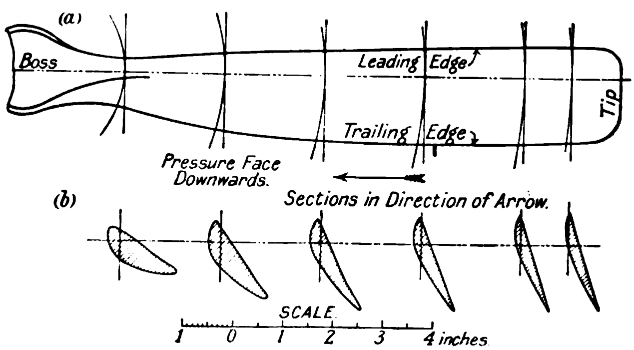


FIG. 109

§ 4. In general the motion of any portion of a propeller moving in still air along the direction of its axis is compounded of the direct translational speed of the shaft to which it is attached, and its own rotational speed. The thrust consequently, or force developed along the axis, will depend on both these components. As far as an experimental test is concerned these may be varied independently, and accurate knowledge of the performance of a propeller would demand information as to how the thrust and torque developed vary with each of these factors.

The normal method of testing is to measure the variation of thrust and torque with the speed of revolution for a series of translational speeds.

Fig. 110 gives thrust and torque required to drive the propeller for a "static test," i.e. when the forward speed is zero, from which it will be seen that the thrust and torque vary very

approximately as the square of the speed. Such a static test, however, does not correspond to any state of affairs in actual flight and its utility is not in any sense practical but is purely of theoretical interest.

Before it is possible to interpret completely the results obtained by tests indicating the performance of an airscrew it will be necessary to discuss certain properties of a propeller in terms of which it will be found convenient to express the performance curves.

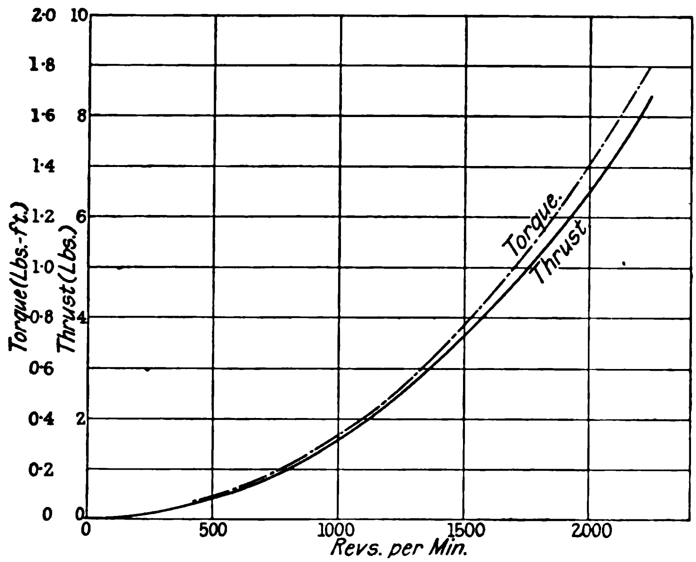


Fig. 110

§ 5. *Pitch.* In many respects a propeller may be looked upon as a screw and the similarity will become clearer as the subject developes. There is however one essential property of a mechanical screw that finds no immediate counterpart in the propeller. In mechanics, the helix of a screw being given, the speed of rotation definitely fixes the forward motion. If V ft. per sec. be the translational speed when the angular rotation is n turns per second, V/n, the distance moved per turn, will be definitely fixed for a given screw, and is normally termed the pitch. With the propeller on the other hand rotating in a medium such as air, the translational and rotational speeds may be varied

independently, and there is no place for a corresponding unique definition of pitch. V/n however will actually measure the pitch of the helix traced out in space by any point in a blade moving forward at V ft. per sec. and rotating at n revs. per sec. For each value of V and n there will exist a definite thrust. In seeking for a definition of pitch applicable to the case of an airscrew, it will be advantageous to consider how the thrust varies with the ratio V/n at different translational speeds.

Fig. 111 represents a series of such curves obtained from tests on a propeller described in the *Reports of the Advisory Committee*

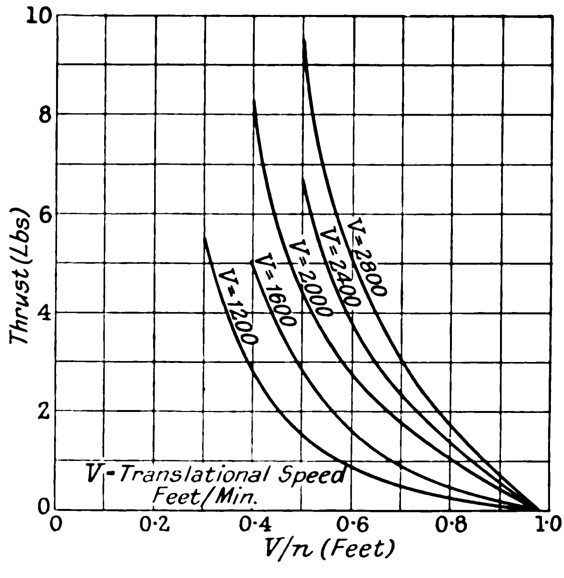


FIG. 111

for Aeronautics, 1911–12 (p. 154). It becomes at once apparent that these curves possess a remarkable property, viz. that corresponding to a value of $V/n = 0{\cdot}98$ ft., the thrust is zero for *all* translational speeds. Since V/n as already explained represents the pitch of the helix traced out in space by, say, the blade tip, 0·98 will be the greatest pitch possible not involving a negative thrust. It will be the same for all speeds, and must accordingly be a constant for that propeller. It is upon the experimental fact that all such curves for a given propeller are concurrent at the point of zero thrust, that the aerodynamic definition of pitch is

based. The pitch p of a propeller is defined as the value of V/n corresponding to zero thrust. Experimentally this definition is found to be unique.

§ **6.** *Slip.* Since V/n measures the distance moved forward per revolution, p will represent the maximum distance that could possibly be traversed during one revolution without developing a negative thrust. The amount by which the propeller has fallen short of this maximum or "slipped" will be given by $p - V/n$. The ratio of this quantity to the maximum possible, namely p, is defined as the slip ratio s.

$$s = (p - V/n)/p$$
$$= 1 - V/np \quad \text{........................(1)}.$$

For accurate information regarding the performance of a propeller it would be convenient to have a knowledge of how thrust efficiency, etc., vary with this quantity, and for this reason the results of such an experimental test are frequently plotted against it as a base. The base might equally well have been chosen as V/n but the advantage that slip ratio possesses in being non-dimensional and therefore independent of the system of units would thereby be lost.

An alternative convenient base for practical purposes is also V/ln where l is the diameter of the propeller. This is also non-dimensional and clearly does not differ essentially from the slip base.

§ **7.** *Thrust and Torque Coefficients.* In practice it is not usual to plot the variation of thrust and torque against slip but rather the thrust and torque coefficients. If in fig. 111, where the thrust in lbs. of a certain propeller is plotted against V/n for a range of forward velocities, another series of curves be derived representing the variation of thrust with forward velocity for a series of constant values of the revolutions per second, it will be found that the thrust is very approximately proportional to the square of the forward speed. Referring back to the discussion on dynamical similarity in Chapter IV, it was there seen that the force on a body may be represented as

$$R = \rho V^2 l^2 f(Vl/\nu).$$

The interpretation of the foregoing curves in the light of this equation is that for all practical purposes $f(Vl/\nu)$ is a constant, which is here defined as the thrust coefficient

$$a = T/\rho l^2 V^2 \quad\text{.......................(2)},$$

l being the diameter of the propeller. It is thus a non-dimensional coefficient, independent of the size of the blade, the velocity of translation, and the density of the air. This question will be returned to later when the whole subject of transition from model to full scale for a propeller will be discussed. For these reasons the variation of thrust coefficient with slip is usually given for the results of tests. In the same way

$$b = Q/\rho V^2 l^3 \quad\text{.......................(3)}$$

is plotted instead of the torque Q.

§ **8.** *Efficiency.* In order to drive the machine through the air, energy must continually be expended to overcome the resisting forces. This must be derived from the engine, and is represented there by a continuous consumption of fuel. If T be the thrust of the propeller in lbs., and equal to the force necessary to overcome the resistance of the aeroplane in steady flight, then TV, where V is the velocity of the machine in ft. per sec., will represent the work done per second against these resisting forces. $TV/550$ will be the horse-power available from the propeller. On the other hand, if Q be the torque and n the number of revolutions per second, $2\pi Qn/550$ will represent the horse-power required to drive the propeller. Taking as the definition of efficiency the ratio between the useful work obtained and the work required to obtain it, the efficiency in this case reduces to $TV/2\pi Qn = \eta$, or in terms of the thrust and torque coefficients a and b

$$\eta = \frac{aV}{2\pi bln} = \frac{1}{2\pi}\cdot\frac{a}{bl}\cdot\frac{V}{n} = \frac{p}{2\pi l}\cdot\frac{a}{b}(1-s) \quad\text{........(4)}.$$

§ **9.** Since this expression for efficiency is likewise non-dimensional, it will be possible to express the whole performance of the propeller by means of curves representing the variation of the three non-dimensional quantities a, b and η with a fourth s, likewise non-dimensional, all of which are accordingly independent of the system of units adopted.

Fig. 112 shows the variation of thrust and torque coefficients and efficiency with slip for a test on a propeller whose plan form and sections are given in fig. 109. It will be seen that the efficiency reaches a maximum at comparatively low values of the slip, an indication of which fact is given in equation (4). At such a low value of the slip, however, the thrust is small and therefore in practice it is advisable to sacrifice efficiency for the sake of extra thrust by working at slightly higher values of the slip.

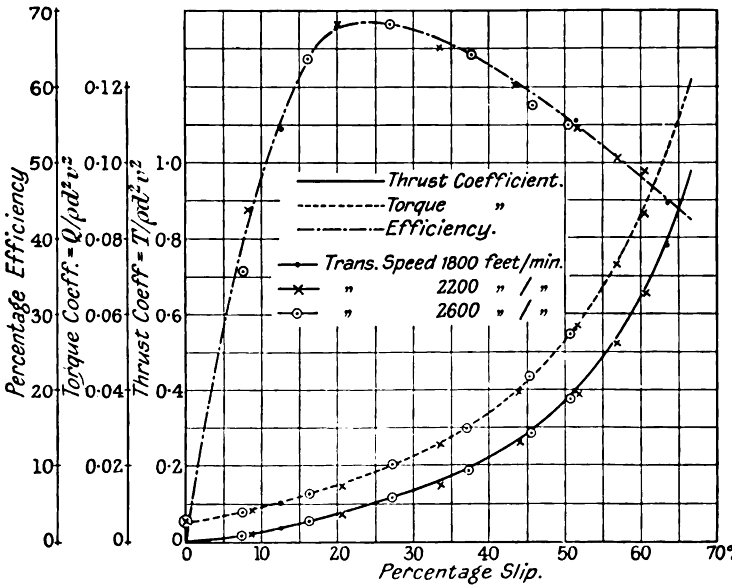


FIG. 112

§ **10.** A number of questions that arise in connection with the propeller and its relation to other parts in the design of a machine may be here briefly referred to.

(*a*) From a knowledge of the requirements for which a machine has been designed and a set of curves representing the performance of a series of propellers, it should be possible to determine which one is most suitable for the purposes of the machine.

Generally it may be supposed that the machine apart from the propeller and engine is given. This might be taken as equivalent for the present purpose to an assumption of the value

of the forward speed of flight and a curve representing the drag on the machine at various speeds. If the position of the axis of the propeller is also fixed a limitation would thus be imposed on the diameter, whose value shall be supposed given.

The thrust at any speed being equal to the drag on the remainder of the machine the variation in thrust coefficient with speed is at once determined, fig. 113 *A*. The propeller most efficient at the required speed of flight may now be selected from a series, from a knowledge of the curves for each, showing the variation in thrust coefficient and efficiency with slip.

Fig. 100 *B* represents the performance curves for three propellers. The normal speed of flight v being given, the corresponding thrust coefficient a is obtained in fig. 113 *A*. Turning to the performance curves, the slip and efficiency of each propeller

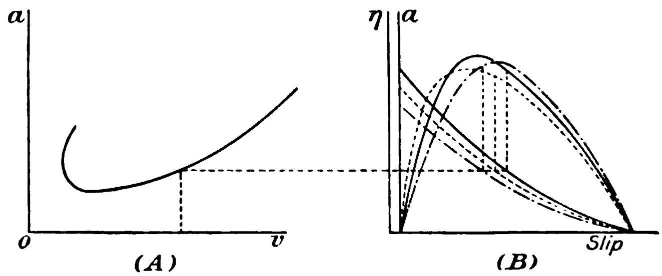


FIG. 113

when its thrust coefficient is equal to this moment are measured. The propeller with the maximum efficiency at this velocity should be the one chosen. If desired the engine revs. can be calculated from the slip. The foregoing remarks must be qualified in so far as requirements of strength may enter to limit the choice.

(*b*) As a further illustration the curve used in Chapter V, § 4, giving maximum horse-power available to drive the machine at various velocities, will be obtained from a knowledge of the brake horse-power curve of the engine and the diameter of the propeller. A test on the engine will give the variation of maximum horse-power the engine can furnish with revs. of engine, and hence, fig. 114, with propeller revs. from a knowledge of the gearing. From the torque curve, fig. 112, the horse-power required to drive the propeller at various translational speeds can be calculated and plotted against revs. Where this system of curves intersects

the B.H.P. curve of the engine the maximum horse-power available at that translational speed to drive the propeller at the corresponding revs. is determined (curve $AA'A_1$, fig. 114). From this, knowing the propeller efficiencies, the required curve, viz. maximum horse-power available against forward speed, is at once derived.

(c) The diagram used in the last illustration is specially convenient for estimating the maximum horse-power available for climb at each speed. Curve ABA_1 represents the horse-climb required to drive the machine at various speeds. This has been obtained by a simple calculation from the knowledge of the resistance of the machine. The ordinate at B gives the horse-power used up in driving the aeroplane at the speed indicated on the curve OB. The ordinate at A' represents the maximum horse-power available at that speed, and the difference between these two ordinates gives a measure of the horse-power available for climb under these conditions. A_1 corresponds to the maximum speed possible for horizontal flight.

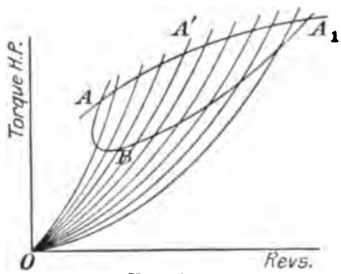


FIG. 114

§ **11.** By means of the considerations developed in the last few paragraphs it is comparatively easy, from a knowledge of the performance of a set of propellers supposed tested, to select that particular one best adapted to fulfil the requirements. On the other hand from a knowledge of the requirements only, no indication has so far been given as to how to design the propeller. As in the case of aerofoil design the problem thus presented can be best answered by an extensive and elaborate series of experiments on the effect on performance of such changes in the propeller as variation in plan form, in breadth of blade, in thickness, and position of maximum camber of the section and in inclination of chord section to the propeller disc, etc. Such information, moreover, as can be derived from a consideration of the aerodynamic properties of a propeller, the nature of the flow of air in its vicinity, the pressure distribution over the blade, and the position of the centre of pressure, would be of extreme value. Unfortunately the experimental study of aeronautics is not

sufficiently old to have given time yet for such an extensive research. In the *Reports of the Advisory Committee for Aeronautics*, 1911–12 (p. 144), a number of tests are described showing the effect on efficiency and thrust, of increasing the blade width for a certain propeller. It is there found that the propeller rapidly increases in efficiency as the blade width increases up to a certain point beyond which the efficiency and thrust (at a constant translational speed) remain fairly constant.

The results of these tests are doubtless of considerable value, but it must be borne in mind that there are so many other factors admitting possibility of variation that this isolated investigation throws little light on the question as a whole.

The subject will be much clarified by a consideration of the propeller from a standpoint already referred to, viz. that it may be looked upon as a continuous series of elementary aerofoils of various forms of section. Should this theory be supported by experiment it will then be possible to utilise the available knowledge regarding aerofoils when the question of design of propeller blades arises. It must be clearly understood that each of the elements of which the blade is supposed constituted must as far as aerodynamic properties are concerned be considered as a very narrow section of an aerofoil of extremely large aspect ratio. The forces on the elementary portion of the blade will then be the force unit per length of the aerofoil multiplied by the length of the elementary section. These forces are supposed to be such as would originate when the aerofoil in question is moving in a straight line with a constant speed equal to that which the section of the blade would have at the moment when the latter is rotating, and with the same attitude to the relative wind as that section.

It does not require much consideration to recognise that in many respects the propeller section exists in a very different environment from that which surrounds the corresponding aerofoil section. Whether these differences are so great as to vitiate the assumption that the forces on the two sections are identical must be tested by experiment. In the first place since the cross-section of the blade varies from point to point the flow over an element cannot be the same as in the case of the infinite aerofoil. The forces must accordingly differ. In the second place an element of a propeller does not move in a straight line into still air but in a helix into air already disturbed by the previous motion of the

propeller, so that the flow of the air and consequently the forces brought into play must be modified. Finally the tip of the propeller at which point the speed is greatest is constituted of an element with a neighbouring element only on one side. A condition of end flow similar to what originates at the tip of an aeroplane wing will thus be produced, which does not correspond to a section in the middle of an aeroplane. On account of these discrepancies it is extremely important to carry out a series of tests in order to determine how far this theory of the constitution of propeller blades is modified by these various forms of interference.

§ **12.** Let fig. 115 represent a section of the blade at radius r from the axis of the propeller making n revs. per sec., while its translational speed along the axis is V ft. per sec. The total distance moved by the section in one second round the axis is $2\pi rn$. V and $2\pi rn$ are therefore the two components of the speed

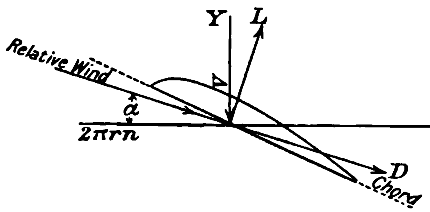


FIG. 115

of the section along the axis and in the plane perpendicular to it, and accordingly the final velocity is $\sqrt{[V^2 + (2\pi rn)^2]}$. This is for the moment supposed to be the speed of the relative wind in the direction indicated in the diagram. If L and D be the lift and drag on the element, the components of the resultant force along and perpendicular to the axis will be

$$L \cos \alpha - D \sin \alpha = Y,$$

$$L \sin \alpha + D \cos \alpha = X,$$

where $$\tan \alpha = V/2\pi rn.$$

Since V is the useful speed developed,

$$(L \cos \alpha - D \sin \alpha)\, V$$

is a measure of the useful work performed per second by the element while $$(L \sin \alpha + D \cos \alpha)\, 2\pi rn$$

represents the work expended in producing this. Taking the efficiency η of the element as the ratio of these two quantities,

$$\eta = \frac{V}{2\pi rn} \frac{(L \cos \alpha - D \sin \alpha)}{(L \sin \alpha + D \cos \alpha)}$$

$$= \frac{V}{2\pi rn} \cdot \frac{\cos \phi \cos \alpha - \sin \phi \sin \alpha}{\cos \phi \sin \alpha + \sin \phi \cos \alpha}$$

$$= \frac{V}{2\pi rn} \cot (\phi + \alpha) \quad \text{.................................(5)},$$

where $\tan \phi = D/L$ ....(6).

For a particular element V, r, n and therefore α are all fixed, and consequently the *element* will have the maximum efficiency when $\cot (\phi + \alpha)$ is a maximum. This occurs here when $\phi + \alpha$ is a minimum, i.e. when ϕ is a minimum. This demands that in order that each section should attain maximum efficiency, its shape should be such as to give a high value of L/D and it should be set at that angle of attack which corresponds to the maximum value of that ratio. Aerofoil sections therefore suitable for propeller design should be such as to furnish a large value of maximum L/D, apart from requirements of strength. It need scarcely be pointed out that maximum efficiency for each *element* does not necessarily correspond to maximum efficiency for the *whole blade*. For comparing two elements, one having high and one having low maximum efficiency, the total efficiency of the two regarded as a whole, each working at its maximum efficiency, could be increased by devoting some of the energy expended on the lower efficiency element to the higher one, thereby gaining in thrust for the same expenditure of work. The loss in thrust on the lower efficient element working at a still lower efficiency is amply compensated for by the gain in thrust from the higher. Although this is so, however, setting each element of a propeller blade at the position of maximum L/D can be taken as a general guide and starting point.

§ 13. The accuracy of this theory was subjected to a severe test by a series of experiments conducted at the N. P. L. and described in the *Aeronautical Journal* for April, 1914 (p. 201). A two-bladed propeller 2′ diameter was tested for thrust, torque and efficiency on the whirling arm, according to the usual fashion. A series of aerofoils were made similar in section to those of the

propeller blade every two inches along its length. This gave six sections in all, and in order to eliminate the effect of aspect ratio as far as possible each aerofoil was made of the same length as the blade, viz. one foot, and equal in breadth to that of the corresponding cross-section. Measurements were taken of the lift and drag forces for various angles of attack for a wind speed of 30 ft. per sec. The actual inclinations of the blades at the

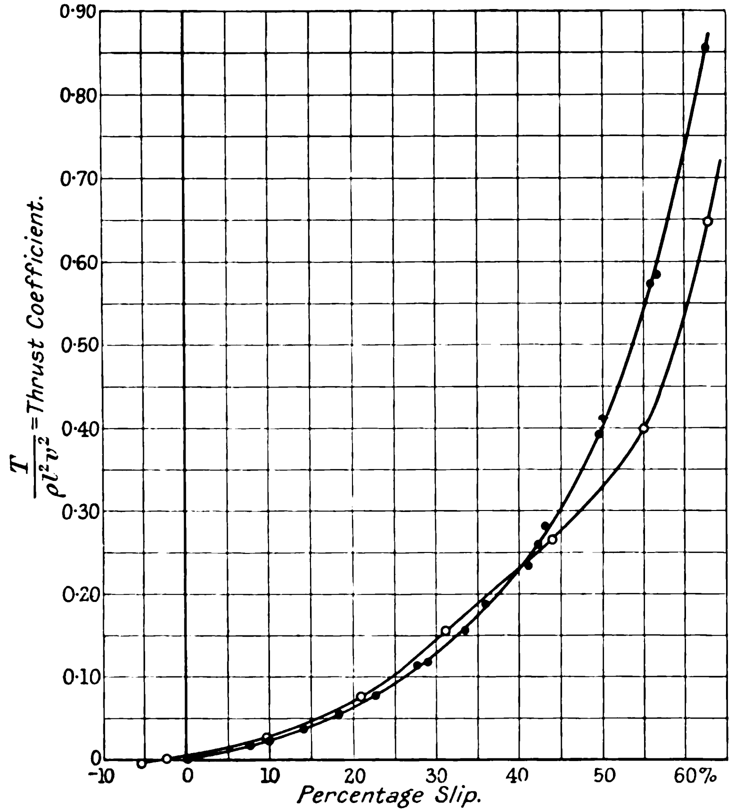


FIG. 116

various radii were accurately measured up, and the forces per unit length on each aerofoil at its angle of attack determined. By integration, the thrust and torque coefficients and efficiencies were calculated and plotted against slip in the same diagram as the experimental values obtained by the propeller tests. These are given in figs. 116, 117 and 118.

Although the tests on the aerofoils were conducted at a speed

of 30 ft. per sec., it was assumed that the lifts and drags varied according to the square law over the range of speed considered. This would not be accurately correct, however, and would necessarily introduce errors. Nevertheless a comparison between the actual and the calculated curves shows that in spite of all the possible discrepancies already pointed out very fair agreement is obtained. This in itself is a remarkable tribute to the aerofoil element theory of propeller blades. In the complete account of the experiments as described more fully in the *Reports of the Advisory Committee*, 1912–13, additional comparisons were carried through, but, although it is evident that the theory cannot be

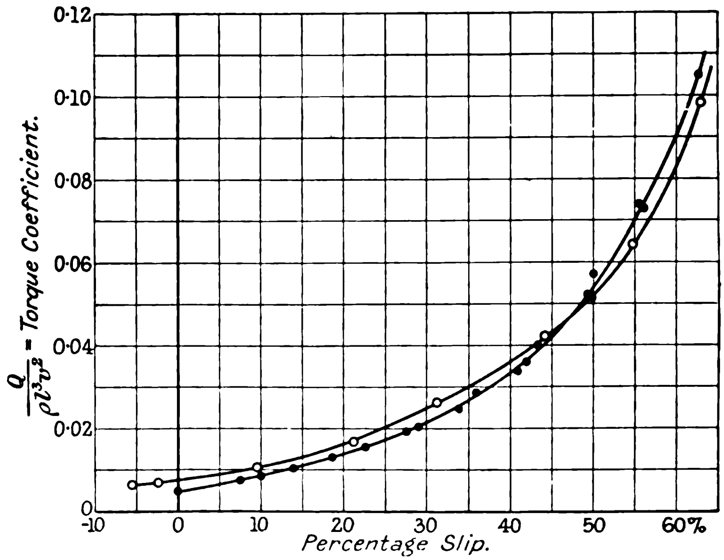


FIG. 117

supposed to hold rigorously, enough has been said to show that if proper allowance be made for the differences that must inevitably arise, the method may be used with considerable accuracy, as a general guide in design.

NATURE OF THE AIR FLOW IN THE NEIGHBOURHOOD OF A PROPELLER

§ **14.** After a discussion of the disturbing effect on the air of the motion of an aerofoil, it becomes perfectly clear that the much more complex case of the rotation of a propeller is far beyond the scope of mathematical analysis as it at present stands. Never-

theless, a general consideration of the effects most likely to be produced can be based on elementary dynamical principles, and much useful information can be derived therefrom. As in the case of the simple aerofoil each propeller blade in its motion may naturally be expected so to disturb the air in its neighbourhood, especially at the high speeds associated with such a body as to produce continually two systems of vortices, one starting

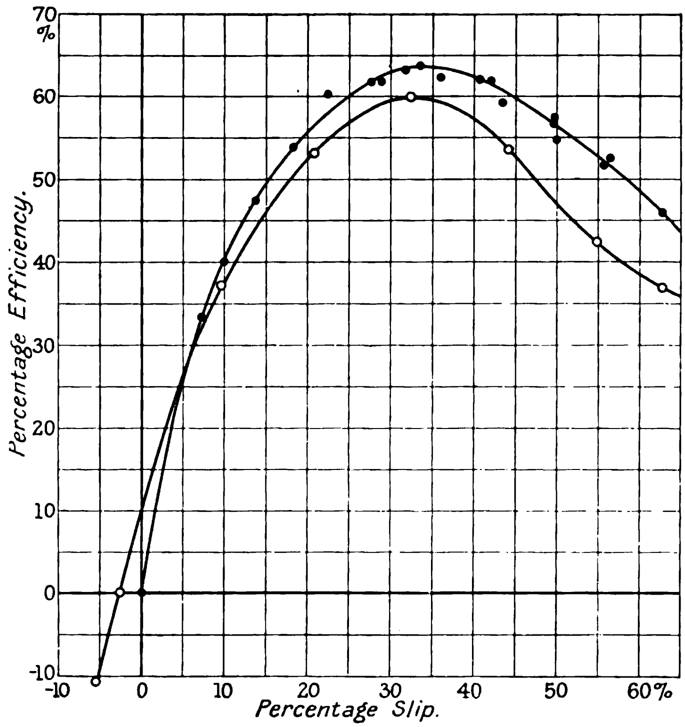


FIG. 118

off approximately from each face. For every element the nature of the origin of these eddies will not differ considerably from that explained for an aerofoil in Chapter II. The problem, however, involves a still further complication from the fact that the blade does not move with constant linear velocity but describes a screw or helical surface in the fluid. The air approaching the propeller will, moreover, also be disturbed before actually passing through the disc of the screw, the effect of which in general will be to

impose an acceleration upon the approaching elements of the fluid until they are finally sucked through the disc and discharged in a state of violent turbulent motion into the slip stream. Even those portions of the air which do not come into the immediate vicinity of the blades but lie at a considerable distance from it are themselves disturbed and attain both a translational and a rotational motion they would not otherwise possess. It is from the propeller that all this energy and momentum in the fluid extending to infinity in all directions is derived, and by estimating accurately the total magnitude and direction of this momentum both translational and rotational and the total quantity of energy imparted to the fluid per second, the thrust, torque and horse-power delivered up by the propeller can be derived. It is clear that until much more accurate information is to hand regarding the exact nature of the flow in all its details the complete problem is beyond the power of the analyst to solve. Permitting certain limitations and assumptions however not necessarily dynamically unsound, it is possible to progress some short distance.

§ **15.** Attempts to investigate this problem are usually based on the assumption that the fluid acted upon is incompressible and non-viscous. This is not essentially illegitimate in view of the fact that the speeds attained by the elements of the blade are in general considerably below that of the velocity of sound so that compressibility effects do not appear. The extent to which the results are vitiated by the neglect of the viscous forces will be considered later when the problem of scale effect in a propeller is analysed. It is sufficient for the present to state that these assumptions are good enough as approximations.

It will be supposed initially that the motion is everywhere symmetrical about the axis of the propeller and that rotational velocities about that axis do not arise. In effect we replace the propeller by a disc more or less extended axially, through which the fluid in front is sucked and discharged behind by the exertion of a thrust operating throughout the whole region of this propeller or actuator disc. It must be clearly understood that only in the sense that the fluid passing through the actuator comes under the influence of a thrust at the instant when it exists there, can the present problem be supposed to deal with the propeller at all.

For convenience it will be supposed that instead of the propeller in motion through the fluid the former is at rest and the fluid moves past it with equal and opposite velocity. This cannot affect the physical facts of the problem but will nevertheless be a much more convenient method of approach.

§ **16.** Hydrodynamically, the problem can be stated as follows. Fluid in motion at infinity with constant velocity V horizontally is acted upon by an external horizontal force T, distributed in a given manner throughout a limited region in space—the actuator—containing the fixed axes of reference. The problem is to determine the nature of the stream-lines and the velocities at all points of the field.

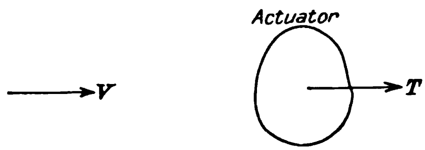


Fig. 119

The two fundamental equations of hydrodynamics applicable to the present problem of a perfect fluid are:

(*a*) the equation of continuity,

(*b*) the pressure equation.

The actual equation of continuity will not be utilised in its ordinary form but the fact that the fluid is continuous and incompressible will be frequently implicit in the equations derived.

At the very outset as far as (*b*) is concerned a vital distinction exists between those portions of the fluid which have passed through the actuator and those which have not. For the former the pressure at any point is given by the pressure equation in its ordinary form, the motion being steady, viz.

$$p = p_0 + \frac{\rho}{2}(V^2 - q_1^2) \quad \text{.....................(7)},$$

where q_1 is the velocity at the point.

On account of the existence of a thrusting force acting upon the fluid within the actuator, fluid which is discharged from it will do so at an increased dynamic pressure. As a consequence, for such portions of the fluid the pressure equation takes the extended form

$$p = p_0 + \rho\Omega + \frac{\rho}{2}(V^2 - q_2^2) \quad \text{..............(8)},$$

where q_2 is the velocity at the point and Ω depends for its magnitude on the particular stream-line upon which the selected point lies.

§ **17.** Let two points be taken one on each side of, but infinitely close to, the stream-line which having passed through the tip of the actuator separates that fluid which has crossed it from that which has not. Since the total pressures at these two points must be equal, the two above expressions indicate that the velocities at these points must differ by a finite amount. It follows that this surface of separation must also constitute a surface of discontinuity in velocity. The air which passes through the actuator is thus discharged in a slip stream enclosed in a surface whose velocity at every point differs by a finite amount from that at a neighbouring point, just outside the slip stream.

It is not proposed to enter into details of the nature of the flow in the immediate vicinity of the actuator but by the application of a general theorem in pressures and momenta to obtain a relation between the thrust developed, and both the velocity with which the fluid enters the actuator and with which it is discharged in the slip stream at a great distance from the propeller. By this means it will be possible to make an estimate sufficiently accurate for practical purposes of the increased velocity with which the air enters the propeller in terms of the thrust developed and the forward speed of the propeller. This information will be found extremely useful in the question of design.

§ **18.** *General Theorem regarding Momentum and Pressure.* Consider a surface enclosing a portion of the fluid and the actuator, supposed situated at the point A and exerting a thrust T in the direction AB. Let p be the pressure —acting normal to the surface at P and inclined at an angle θ to AB. If the state of motion within this region be assumed steady, then the total force—thrust and pressures acting in the direction AB—is equal to the rate of transference of momentum from within over the surface in this direction to the fluid without.

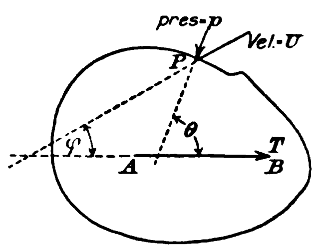


FIG. 120

Hence

$$T - \int_s ds\,(p\cos\theta) = \int_s \rho U^2 ds \cos\phi \cos(\theta - \phi) \quad \text{......(9)},$$

where ϕ is the inclination of the resultant velocity U, at the point P, of the fluid relative to the surface.

§ **19.** *Static Problem.* The case where the propeller is at rest relative to the undisturbed air at infinity will be first discussed. The previous arguments regarding the existence of a discontinuous surface in the wake of the actuator will still apply with equal force, but in front no such definite surface can exist. Imagine the outside surface of the slip stream replaced by a tube of exactly the same shape and dimensions through which the air is sucked by the actuator with exactly the velocities it possessed before the introduction of the tube. Over the surface of a very large sphere of centre A the suction at A along the tube will operate equally in all directions and the flow will consequently be radial and constant. If u be this velocity, then the total mass of air transferred over the surface of the sphere per second is $4\pi R^2 \rho u$ where R is the radius of the sphere. If v be the velocity along the slip stream of radius a at a distance r from the axis, then the fluid discharged per second is

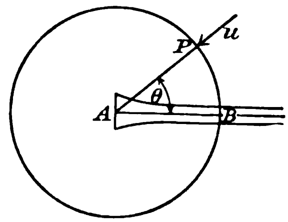


FIG. 121

$$\int_0^a 2\pi r\,dr\,.\,\rho v.$$

Hence $$2R^2 u = \int_0^a rv\,dr = Q \text{ say} \quad \text{....................(10)}.$$

Let AP be the direction of the stream-line at infinity which is continued along the slip stream at radius r from the centre and let its inclination to AB be θ, then the total mass of fluid entering the sector of the sphere obtained by rotating AP about AB is

$$2\pi R^2 \rho (1 - \cos\theta)\, u = \int_r^a 2\pi r\,dr\,v\rho.$$

Hence $$1 - \cos\theta = \int_r^a \frac{2rv\,dr}{Q} = 2 - \int_0^r \frac{2rv\,dr}{Q},$$

giving $$\cos^2 \theta/2 = \frac{1}{Q}\int_0^r rv\,dr \quad \text{.................(11)},$$

which expresses θ as a function of r.

§ **20.** *Integration of Pressure over Sphere.* Since the flow is everywhere radial the horizontal components of the pressures on the outside surface of the sphere are in equilibrium.

§ **21.** *Transference of Momentum over Sphere and Slip Stream.* Regarding the sphere as a whole as much momentum evidently enters as leaves provided the flow be supposed to take place with velocity u along the slip stream towards the centre of the sphere. The correction for this is clearly of the fourth order of small quantities. Momentum leaving along the slip stream

$$= \int_0^a 2\pi r\,dr\rho v^2.$$

Hence, since the total force in the horizontal direction is T, this gives
$$T = \int_0^a 2\pi\rho\, rv^2 dr \text{........................(12)}.$$

If v_1 be the horizontal component of the velocity at radius r at the mouth of the tube, i.e., the inflow velocity at the propeller disc, then the total quantity of fluid entering is

$$\int_0^{a_1} 2\pi r\,dr\,\rho v_1 = 2\pi\rho Q \quad \text{..................(13)},$$

the total mass of fluid passing down the slip stream, where

$$Q = \int_0^a rv\,dr \quad \text{........................(14)}.$$

§ **22.** If the propeller be of such a nature as to provide a constant distribution of velocity across the section of the slip stream, then this expression for the thrust becomes

$$T = A\rho v^2 \text{..........................(15)},$$

where A is the area of the slip stream. If v_1' be the average velocity of inflow, then

$$\pi\rho a_1^2 v_1' = 2\pi\rho Q = \pi\rho a^2 v.$$

The average velocity of inflow to outflow is thus

$$\frac{v_1'}{v} = \frac{a^2}{a_1^2} = \frac{\text{area of cross-section of slip stream}}{\text{area of propeller disc}},$$

giving $$T = \rho A_1^2 v_1^2 / A = \rho K A_1 v_1^2 \quad \text{.................(16)},$$

where A_1 is the area of the propeller and $1/K$ is the contraction, i.e. the ratio of the area of the slip stream at an infinite distance behind the propeller to the area of the propeller.

§ **23.** *Case of a Propeller moving with Translational Velocity V relative to Still Air.* As already stated the problem will be approached by the more convenient method of supposing the propeller stationary and the air moving past with velocity V at infinity. It will be assumed, although rigorously further proof is required, that the velocity at a great distance R from the propeller at any instant is compounded of a horizontal component V and a uniform velocity u directed towards the centre of the propeller. This for the moment may be considered tantamount to restricting the nature of the propeller.

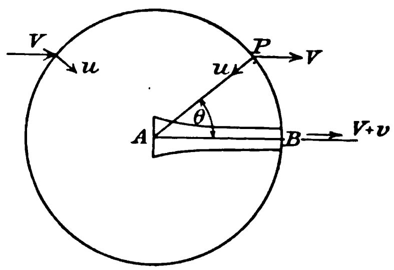


FIG. 122

Accordingly the total transference per second of matter across the sphere is $4\pi R^2 u\rho + \pi a^2 \rho V$ excluding slip stream. The amount leaving per second through the slip stream is

$$\int_0^a 2\pi r \rho \, dr \, (V + v).$$

Equating these

$$4\pi R^2 u\rho + \pi a^2 \rho V = \pi a^2 \rho V + 2\pi\rho \int_0^a rv\,dr,$$

therefore $$u = \frac{1}{2R^2}\int_0^a rv\,dr = \frac{Q}{2R^2} \quad \text{.................(17)}.$$

§ **24.** *Integration of Pressure over Sphere.* The velocity at the point P is $\sqrt{(u^2 + v^2 - 2uV\cos\theta)}$ and accordingly the pressure normal to the sphere is

$$p_0 + \frac{\rho}{2}\{V^2 - (v^2 - 2uV\cos\theta + u^2)\} = p_0 + \rho u V \cos\theta,$$

neglecting second order terms. The horizontal component of this in the direction AB is $(p_0 + \rho u V \cos\theta)\cos\theta$ and this integrated over the whole sphere becomes

$$-\int_0^{\pi} 2\pi R \sin\theta \,.\, R d\theta\,(p_0 + \rho u V \cos\theta)\cos\theta$$

$$= -\int_0^{\pi} 2\pi R^2 . \rho u V \cos^2\theta \sin\theta\, d\theta$$

$$= -\tfrac{4}{3}\pi R^2 u V\rho = -\tfrac{4}{3}\pi V\rho . Q \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(18),$$

which is a finite quantity.

§ **25.** *Transference of Momentum over Sphere and Slip Stream.* The elementary annulus passing through P and perpendicular to AB is of area $2\pi R \sin\theta . R d\theta$; and hence the mass of fluid leaving the annulus per second is

$$2\pi R \sin\theta . R d\theta \,\{V\cos\theta - u\}\,\rho,$$

since $(V\cos\theta - u)$ is the total radial velocity. The component motion in the direction AB is $V - u\cos\theta$ and hence the momentum transferred per second over the annulus in that direction is

$$2\pi R \sin\theta . R d\theta . (V\cos\theta - u)\,\rho\,(V - u\cos\theta).$$

Integrating this over the whole sphere where θ ranges from π to 0, this becomes

$$-\tfrac{8}{3}\pi\rho V Q.$$

To this must be added $-\pi\rho a^2 (V+u)^2$ since the above calculation has supposed this amount transferred through the slip stream. The momentum issuing across the section at B in the slip stream is

$$\int_0^a 2\pi r\, dr \rho\,(V+v)^2 - \pi a^2 \rho V^2 + 4\pi\rho V\int_0^a v r\, dr + 2\pi\rho\int_0^a v^2 r\, dr.$$

Hence the total momentum transferred per second over the whole system in the direction AB is

$$-\tfrac{8}{3}\pi\rho V Q - \pi\rho a^2 V^2 + \pi\rho a^2 V^2 + 4\pi\rho V Q + 2\pi\rho\int_0^a v^2 r\, dr$$

$$= \tfrac{4}{3}\pi\rho V Q + 2\pi\rho\int_0^a v^2 r\, dr \ldots\ldots\ldots\ldots\ldots\ldots(19).$$

Equating this to the forces acting in the direction AB on the system

$$T - \tfrac{2}{3}\pi V\rho Q = \tfrac{4}{3}\pi\rho VQ + 2\pi\rho\int_0^a v^2 r\,dr.$$

$$\therefore T = 2\pi\rho VQ + 2\pi\rho\int_0^a v^2 r\,dr \quad \text{............(20).}$$

§ **26.** *Example* 1. Suppose the velocity distribution in the slip stream is parabolic in v across the radius, vanishing at $r = 0$ and $r = a$, then

$$v = \frac{12Q}{a^4}(a - r)\,r,$$

which as can easily be verified satisfies the equation

$$T = 2\pi\rho VQ + 2\pi\rho\,.\frac{144Q^2}{a^8}\int_0^a (a - r)^2 r^3 dr.$$

$$\therefore\ T = 2\pi\rho V\frac{a^2\bar{v}}{2} + 1{\cdot}2\pi\rho a^2\bar{v}^2,$$

where $\bar{v}$ is the average velocity of outflow.

Hence
$$\bar{v} = \frac{10V}{24}\left[\sqrt{\left(1 + \frac{4{\cdot}8T}{\pi\rho a^2 V^2}\right)} - 1\right]$$
$$= {\cdot}42V\,[\sqrt{(1 + 4{\cdot}8K\alpha)} - 1],$$

where $K = r_0^2/a^2$ and $\alpha = T/\pi\rho r_0^2 V^2$.

Example 2. Suppose the velocity in the slip stream is constant over the section, then

$$T = 2\pi\rho VQ + \pi\rho v^2 a^2,$$

where
$$Q = \int_0^a vr\,dr = va^2/2.$$

Hence
$$v^2 + vV - T/\pi\rho a^2 = 0,$$

giving
$$v = {\cdot}5V\,[\sqrt{(1 + 4K\alpha)} - 1].$$

Taking $\alpha = 0{\cdot}5$ and the contraction as 0·65, this gives in example 1
$$\bar{v}/V = 0{\cdot}51,$$
and in example 2
$$v/V = 0{\cdot}53,$$
$$\therefore\ v - \bar{v} = 0{\cdot}02V.$$

This provides a measure of the error involved in assuming, in the first case where the velocity distribution was parabolic and the contraction 0·65, that the velocity was uniform across the section. Such a discrepancy in velocity is negligible in this case. 0·65 may in general be taken as an average value for the contraction.

§ 27. **As in the static case the average inflow velocity can be obtained by equating the quantity of fluid passing across the propeller disc to that passing down the slip stream per second, either assuming the inflow constant, or varying across the disc according to any empiric law of distribution. The details of this may be left as an exercise, but it may be here stated that if the distribution of the additional inflow velocity over the propeller disc be assumed uniform, the conception that the rate of discharge of energy into the fluid is maintained by the work done upon the fluid by the propeller thrust alone is sufficient to determine, along with the expression for the thrust already derived, that the additional inflow velocity is equal to one-half the additional outflow velocity. For a discussion of this point the student is referred to papers by Froude in *Arch. Inst. Nav. Arch.* 1889 and 1911.**

Assuming therefore that the velocity of inflow, assumed uniform over the propeller disc, is given to sufficient accuracy by half the outflow velocity, and since the latter can be calculated closely enough by the foregoing methods in terms of the thrust the propeller is required to furnish, it is no difficult matter to apply these results immediately in the question of design. The thrust is of course determined by the drag of the rest of the machine at the given speed. Instead of each blade element meeting the air with velocity

$$\sqrt{[V^2 + (2\pi rn)^2]},$$

as given in § 12, the additional inflow velocity merely requires to be superposed, and the resultant velocity takes the amended form

$$\sqrt{[(V + v/2)^2 + (2\pi rn)^2]}.$$

The corrected angle of attack is now given by

$$\tan \alpha = (V + v/2)/2\pi rn.$$

This determines the setting of the blade elements to give maximum efficiency to each. In practice, since the application of the inflow is of the nature of a correction, it is sufficiently accurate to assume it a constant over the propeller disc.

§ 28. *Scale Effect.* In a previous chapter it was explained from a theoretical discussion how the forces originated by the motion of an aerofoil vary with change in the dimensions. Experimental evidence was there adduced amply to support the conclusions arrived at. If the aerofoil element theory of propeller

blades is to be accepted as a satisfactory basis on which to found conclusions regarding the performance and therefore the design of a propeller, it would naturally be expected that corresponding scale effect should be found in this case also. Each element of the blade being a miniature aerofoil will experience its own particular scale effect, and how these sum up undergoing mutual interference must receive special investigation. From our previous information on this subject some such effect is to be anticipated. In one essential however the present case differs from that previously discussed. The translational speed of an aerofoil rarely exceeds 150 ft. per sec.; a propeller blade on the other hand, 8 ft. diameter rotating at 1500 revs. per min., propels its tip through the air at a speed of approximately 650 ft./sec. In the case of the aerofoil it was assumed that the only properties of the air affecting the forces were the density and viscosity. This is amply justified for such comparatively low speeds by experiment, but for high speed projectiles it has been shown that when the velocity approximates to that of sound, another factor, namely the compressibility, comes into operation.

Although, as already explained in Chapter II, the sharp rise in the resistance coefficient takes place in the immediate neighbourhood of the velocity of sound, compressibility effects make themselves evident at much lower speeds. Without further evidence leading to a contrary conclusion it will be necessary to suppose that the thrust on a propeller may depend, in addition to the viscosity and density, upon the compressibility of the medium. For the present purpose this is best represented by the velocity of sound which depends directly on that quantity. How far both the compressibility and viscosity modify the results, at the ordinary speeds of propellers, will be discussed later.

§ 29. Suppose the propeller is given by a drawing, the diameter of length l, to scale. The properties of the fluid through which the propeller is moving that will affect the thrust developed are the density ρ, the viscosity ν and the velocity of sound in the medium V. The factors of the propeller that determine the thrust developed in this medium are the translational velocity v, the rotational speed n, and the diameter l. These quantities and these alone together with the shape of the propeller are responsible for

the magnitude of the thrust. Following along the lines of Chapter IV, each term in its expression must be represented in the form

$$\rho^{\alpha}\, v^{\beta}\, l^{\gamma}\, n^{\delta}\, V^{\lambda}\, \nu^{\mu} \qquad\qquad (21).$$

A thrust being a force must have dimensions ML/T^2 and the above expression must likewise reduce to this. The dimensions of (21) are

$$\frac{M^{\alpha}}{L^{3\alpha}} \cdot \frac{L^{\beta}}{T^{\beta}} \cdot L^{\gamma} \cdot \frac{1}{T^{\delta}} \cdot \frac{L^{\lambda}}{T^{\lambda}} \cdot \frac{L^{2\mu}}{T^{\mu}} = \frac{M^{\alpha} . L^{\beta+\gamma+\lambda+2\mu-3\alpha}}{T^{\beta+\delta+\lambda+\mu}} = \frac{ML}{T^2}.$$

Equating powers of M, L and T,

$$\alpha = 1,$$
$$\beta + \gamma + \lambda + 2\mu = 4,$$
$$\beta + \delta + \lambda + \mu = 2.$$

Hence

$$\alpha = 1,$$
$$\beta = 2 - \delta - \lambda - \mu,$$
$$\gamma = 2 + \delta - \mu.$$

Expression (21) now takes the form

$$\rho v^2 l^2 \{(v/ln)^{-\delta} . (v/V)^{-\lambda} (vl/\nu)^{-\mu}\}.$$

Each term in the expression for the thrust will therefore have the factor $\rho v^2 l^2$ and will depend in addition merely on the quantities v/ln, v/V and vl/ν.

An equivalent statement of this is

$$T = \rho v^2 l^2 f[V/ln,\ v/V,\ vl/\nu] \qquad\qquad (22).$$

§ **30.** If compressibility and viscosity effects are appreciable to any extent at the speeds of ordinary propellers, this must be the form that the expression for the thrust must take. An attempt to proceed further along the lines previously developed, by running two similar propellers of different diameters in two different media, say air and water, at corresponding speeds of translation and rotation, fails, for this would give

$$v/v_1 = V/V_1,$$
$$l/l_1 = \nu_1 V/\nu V_1,$$
$$n/n_1 = \nu V^2/\nu_1 V_1^2,$$

and

$$T/T_1 = \rho \nu_1^2 V^4/\rho_1 \nu^2 V_1^4.$$

Remembering that V for air = approx. 1085 ft./sec. and V_1 for water = 4720 ft./sec., $\rho/\rho_1 = {\cdot}00129$, $\nu/\nu_1 = 13$, it will be

easily seen that even if the corresponding translational and rotational speeds were attainable, the thrust developed would be too high for accurate measurement. To proceed further, therefore, it will be necessary to refer back to the equation for the thrust in order to discover what modifying assumptions, if any, may be made regarding the quantities there introduced, without vitiating the correspondence to actual conditions. One might suspect that, at such high speeds, the dragging effect due to the viscosity of the air would be inconsiderable in comparison with the dynamic pressures developed. If the speed, moreover, sufficiently high to make the viscous forces negligible, is yet not too high to bring in compressibility effects to any appreciable extent, it might be legitimate to neglect both the compression term v/V and viscosity term vl/ν in the equation (22). Whether one is entitled to make such assumptions or not is purely a question that must be settled by experimental evidence. If it were so, then equation (22) would become

$$T = \rho v^2 l^2 F\,(v/ln) \quad \text{.....................(23).}$$

§ **31.** In 1906, D. W. Taylor published results obtained by experiments in water on three sets of five similar propellers of different diameters. It was found that for the same value of v/ln the thrusts were proportional to v^2l^2. E. Dorand in 1910, as the result of tests on two similar propellers 2·5 and 4·3 metres in diameter respectively, moving with a translational speed of 25 metres per second, found that the forces on the two propellers likewise varied as the squares of the speed and diameter for the same value of v/ln. This was later borne out in 1911 by Eiffel. A comparison between a propeller two feet in diameter tested on the whirling arm at the N. P. L. and a similar one 15 feet in diameter tested on Messrs Vickers' whirling arm also showed excellent agreement in this respect (1910–11). It would seem therefore that ample evidence exists to conclude that, when v/ln is kept constant, $f(v/ln,\ vl/\nu,\ v/V)$ reduces to a constant. This is equivalent to stating that the terms vl/ν and v/V have no appreciable effect on the value of the expression for the thrust. Experiments therefore justify the statement that viscosity and compressibility effects are negligible. Exactly the same argument applies to the torque and efficiency.

§ 32. It will be remembered that the justification for assuming the existence of a definite constant, pitch, for a given shape of propeller lay in the fact that for all values of forward speed the thrust was found to vanish at the same value of v/ln. This statement is equivalent to saying that there is no scale effect for p/l where p is the pitch and l the diameter.

Now $$s = 1 - v/np = 1 - v/ln \,.\, l/p,$$

therefore $v/ln = (1 - s)\, p/l = \text{constant} \times (1 - s)$ for similar propellers. Equation (23) now reduces to

$$T = \rho v^2 l^2 F_1(s) \quad \text{.......................(24)},$$

and it follows that for the same value of the slip the thrust coefficient $T/\rho v^2 l^2$ is a constant. There is no scale effect for similar propellers at the same value of the slip ratio. This implies that if the thrust coefficient were plotted against slip for similar propellers for all values of v and n, the points thus obtained should all lie on a single curve. The same applies to the torque and efficiency. Fig. 112, giving the results of a series of tests on the same propeller at different speeds, which of course is exactly equivalent to change of scale, brings out the foregoing conclusions.

Equation (24) and the corresponding equation for the torque indicates that the thrust and torque coefficients may be regarded as functions of s only and consequently they may be expanded either in terms of s or what amounts to the same thing, in terms of V/np. The coefficients in these expansions will of course depend on the particular form of propeller blade but on nothing else. Attempts to determine a simple form of expansion of this nature have shown that in practically all cases they may be written

$$a = \frac{4}{3}\left[1 - \left(\frac{V}{np}\right)^2\right]\alpha,$$

$$b = \left[1{\cdot}104 - 0{\cdot}833\left(\frac{V}{np}\right)^3\right]\beta,$$

where α and β are constants and equal in magnitude to the thrust and torque coefficients when $V/np = \frac{1}{2}$. These formulae apply with fairly good accuracy between $V/np = 1{\cdot}0$ and $0{\cdot}5$ and with moderate accuracy from $V/np = 0{\cdot}5$ to $V/np = \text{zero}$.

§ **33.** From the point of view of the element theory this seems a startling conclusion. As already pointed out the existence of scale effect for aerofoils had suggested the possibility of a corresponding effect with propellers and yet it does not appear to exist. A general explanation however is not far to seek. The student will easily verify by a simple calculation and by referring back to Chapter VI that the value of vl for the elements most effective for thrust, viz. about a quarter of the radius from the tip, v being the speed of the element relative to the air and l the length of chord, is already in all such cases sufficiently great to be beyond the range of variation of lift coefficient with vl.

§ **34.** If aerodynamic considerations were the only effective ones that required to be allowed for in propeller design, it is evident that the limitations thus imposed would allow a moderate amount of latitude in the selection of blade sections to satisfy these demands*. There is, however, another aspect of the question which is at least as important as the aerodynamical one on account of the restrictions that are demanded thereby. It has not infrequently occurred in the history of aeronautics that serious accidents have resulted from the snapping of a propeller in mid-air. An analysis of these cases has brought to light several important points. If the relative wind is not directly along the axis then, as the propeller rotates, the thrust on a blade is not constant but varies with its position. It is thus subjected to a periodic force and it is evident that if the natural period of vibration of any portion should correspond to the period of this external impressed force resonance effects will be set up with possible serious consequences. In the second place each element of the blade is rotating at a comparatively high speed round the shaft and the centrifugal forces so produced may reach a considerable value. Unless the requirements of strength so demanded are satisfied the propeller will snap. In addition to the centrifugal forces there exists the direct pressure of the air on each section of the blade and this likewise, because of the moment it produces, requires modifications in the design.

On account of the large number of factors involved no very definite or satisfactory systematic method can be devised for the design of a blade. Ultimately every new propeller is simply

* Cf. § 12.

an intelligent modification of a previous one of known performance, but certain general considerations based on the preceding discussions may be used as a guide. In general the engine revolutions, the maximum speed of flight of the machine and the thrust required from the propeller are previously known. These factors in themselves are of course not sufficient to determine the propeller uniquely even if it were possible practically to do so, but it is desirable to find a propeller with as high an efficiency as possible to satisfy the requirements. From a knowledge of the thrust and the speed of flight the value of the inflow velocity can be approximately estimated, and thus the speed of each element of the blade relative to the air established. Practical experience has shown that the distribution in thrust along the blade or the thrust grading curve as it is called is in general a curve of the type shown in fig. 123 where the area enclosed between it and the horizontal

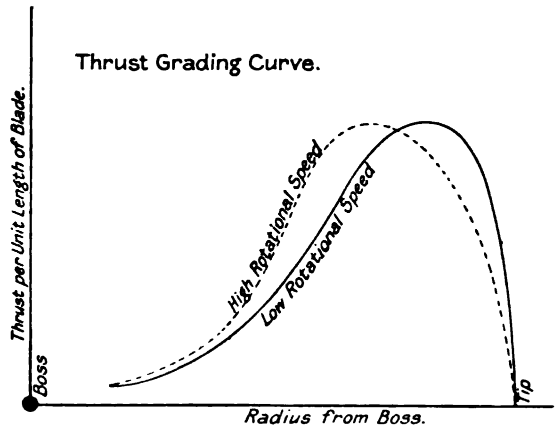


FIG. 123

axis is equal to the given thrust on the blade. A selection of aerofoil sections must now be made such that when set in position at various points along the blade they provide a thrust gradient approximately equal to that given by fig. 123. At the same time the area of cross-section of these elements must be so determined as to provide sufficient strength at each section to withstand the bending moments and forces already referred to. Various such sets of elements may be selected but that set should be chosen which gives greatest efficiency of the combination. The stress calculation with regard to these cross-sectional areas is proceeded

with most conveniently from the tip towards the boss. It may be remarked that if the thrust grading curve be so drawn as to fall slowly towards the tip (see dotted line, fig. 123), the type of propeller corresponding will be one of sharp tip and consequently suitable for high speed work. In the same way a rapid fall towards the tip corresponds to a square tipped blade as required for lower speeds. Experience in design alone can indicate where modifications may be inserted in the resulting blade to remove any bad features that result. One point may be noticed; by a suitable alignment of the centres of gravity of the sections the bending moments due to wind forces may be made to operate in opposition to those due to centrifugal forces. This produces what is usually termed rake in a propeller. The effect of this, and in fact the whole question of stresses, should be accurately worked out on the final propeller for the whole flying range.

§ **35.** It may appear that a considerable amount of the foregoing analysis has involved assumptions more or less arbitrary in character. The fact that the thrust grading curve shown in fig. 123 is in general the type represented depends very largely on the element aerofoil theory, while the latter itself is after all not too well established. Interesting experimental evidence approaching the problem from an entirely new angle is to hand tending to confirm in a large measure much of what has gone before. Some recent experimental work conducted by Dr T. E. Stanton, National Physical Laboratory, indicates that from a measurement of the pressure head at numerous points in the slip stream the general streaming character of the motion here assumed appears to be maintained. Measurements of the increase in pressure head between pairs of points fore and aft of the propeller disc and as close together as is practically possible is taken as an estimate of the thrust per unit length of the propeller blade in that neighbourhood. For a considerable number of propellers for which the experiments were conducted a thrust grading curve was thus derived and the result checked by integration of the curve and comparison with actual thrusts measured. Ample confirmation was thus obtained to indicate that the conditions of the foregoing theory are substantially fulfilled and the thrust grading curve as shown in fig. 123 may be taken as a sufficiently accurate starting point in the design of a blade.

§ **36.** During flight the stresses brought to play upon the blades give rise to a considerable amount of warping so that the blade in operation differs from that as designed, involving a consequent difference from the anticipated performance. A sufficiently accurate estimation of the magnitude of the warping may be made from a knowledge of the forces on each portion of the blade and the position of the centre of pressure on each section. This latter point is usually assumed with sufficient closeness to be the same as that on the corresponding aerofoil of the same section. Permanent twisting of the wood, a comparatively common feature in propellers, cannot be allowed for, of course, and is frequently responsible for great differences from the estimated performances.

In the actual manufacture of blades great care is exercised to ensure that the propeller disc is exactly balanced about its centre, otherwise at the high speeds of rotation serious consequences might ensue on account of the stresses brought into being by vibration. For a high speed propeller if the centre of gravity were in error by about 1/600 of an inch the centrifugal force originated would be about 10 lbs., alternating in direction about 35 times per second.

PART V

CHAPTER XI

STEADY MOTION IN FLIGHT

§ **1.** The motion of a machine will be completely determined by the forces that are brought to bear upon it, and the initial conditions of motion. With the exception of the attraction due to gravity these forces are purely aerodynamic, and as such are determined by the speed of the machine at the moment relative to the air, the shape of its parts, and by its attitude. For a machine at rest there usually exists a vertical plane about which the various parts of the aeroplane are symmetrically distributed. It will shortly become apparent that steady linear flight is always possible without affecting this condition of symmetry which, however, would no longer exist for any type of motion that necessitated say the turning of the rudder, or an alteration in the setting of the wing-flaps. Strictly, a machine which only possessed one propeller would require to have its centre of gravity slightly displaced from the central position, or an equivalent slight alteration in the wing-flaps, to counteract the propeller torque, but for the present this will be neglected. The axis of the machine, as usually understood, is a line in this plane of symmetry through the centre of gravity and parallel to the propeller shaft, but the term is often loosely applied to the horizontal line through the centre of gravity in that plane.

For all practical purposes the propeller thrust may be taken as acting through the centre of gravity although rigorously this is not quite true. The pitching moment brought into existence in the case where this does not hold will be assumed neutralised by an adjustment of the elevator setting as already indicated in Chapter VII. As far as the pilot is concerned, he may be assumed seated in a stationary aeroplane past which the air is streaming. A change in attitude of the machine will present itself to him as an alteration in the direction of the relative wind. When the

conditions in which the pilot finds himself do not alter with time the motion of the aeroplane will be defined as steady. Under these circumstances, once the controls have been placed in position for flight at any instant, no alteration of their setting should be necessary to maintain the motion.

§ **2.** A little consideration will show that this definition of steady motion, involving constantly the same attitude to the direction of gravity and to the relative wind, leaves only three types of flight paths that could, in this sense, be regarded as those of steady motion. These are:

(*a*) Motion in a straight line.

(*b*) Motion in a horizontal circle.

(*c*) Motion in a vertical helix.

A conclusive proof that these are possible flight paths of steady motion in still air would demand that it be shown that it is possible to follow these paths with an aeroplane, controls fixed, moving with constant speed. This will be brought out as the subject develops.

§ **3.** *Motion in a Straight Line. Symmetrical Flight.* It will be assumed that the machine is flying symmetrically with regard to the vertical plane containing the line of flight. This is tantamount to stating that there is no yaw or roll, and that the directions of the rudder and relative wind are the same. The thrust T for all practical purposes may be supposed acting along the flight path and through the centre of gravity of the machine. Since the only other forces beyond this are the weight W, and the resultant air force R whose components are L and D respectively acting perpendicular to and along the flight path, R must also act through the centre of gravity in order that there may be no resultant moment to alter the aspect of the machine. Resolving forces along and perpendicular to the flight path, fig. 124,

$$m\ddot{s} = T - D - W \sin\theta,$$
$$0 = L - W \cos\theta.$$

For steady flight $\ddot{s} = 0$ and hence

$$T = D + W \sin\theta \quad \text{.......................(1)},$$
$$L = W \cos\theta \quad \text{.............................(2)}.$$

For future work it will be convenient to express L and D the total lift and drag on the machine including that on the wings in the form $L = \rho A k_l V^2$ and $D = \rho A k_d V^2$, where A is the area of the wings. k_l and k_d may now be regarded as lift and drag coefficients for the whole machine. Practically k_l differs little from K_l since the wings furnish almost the entire lifting force for the machine. Remembering that

$$L = A k_l \rho V^2,$$

where V is the velocity at the moment, equation (2) then gives

$$V = \sqrt{\left(\frac{W \cos \theta}{A k_l \rho}\right)} \quad \text{.....................(3).}$$

It follows at once that for a given angle of attack, V is constant so long as θ is constant, and conversely. Straight line flight, therefore, involves constant speed of flight. For a given velocity and angle of flight path, taking ρ as constant, this equation likewise determines the lift coefficient and therefore the angle of attack.

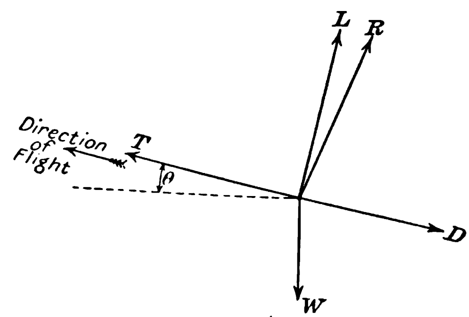


FIG. 124

Corresponding to each angle of attack there is a definite elevator setting in order that there may be no pitching moment on the machine. If θ be small as it usually is except for steep dives, *the speed of flight determines the elevator setting, and vice versa.* A clear realisation of this conclusion will smooth the way towards an understanding of much of the work that follows.

Horizontal flight will of course occur when $\theta = 0$ in which case $V = \sqrt{(W/A k_l \rho)}$. It should be remarked that the landing speed or minimum velocity of horizontal flight is given by this equation when for k_l is substituted the maximum lift coefficient.

§ **4.** *Gliding Flight.* If the propeller be switched off, the machine will commence to glide. The gliding angle will be given by

$$\tan\theta = -\frac{D}{L} \quad \text{........................(4)},$$

obtained from equations (1) and (2) by putting $T = 0$, where D now includes the propeller resistance. In cases of emergency, it is often required to alight when flying over a stretch of ground not suitable for landing. Under such conditions gliding at such an angle as would give the minimum velocity of fall affords the pilot the best opportunity of selecting his landing point. The minimum angle of glide will of course occur when $\tan\theta$ is a minimum, that is at the flying position corresponding to maximum L/D. The velocity along the path will as before be given by equation (3). A machine may therefore glide from any point to earth down any straight line situated within a cone of semi-vertical $\tan^{-1}(L/D)_{\text{max.}}$. Rigorously, however, the angle of glide corresponding to the slowest velocity of drop is not exactly the minimum angle of glide but lies close to that position. The student may easily verify this by determining the value of L/D that makes $V\sin\theta$, the velocity of drop, a minimum, where V is given by equation (3).

§ **5.** *Vertical Dive.* The machine is assumed to be diving vertically under the action of gravity and the thrust of the propeller. Since there must be no forces perpendicular to the line of flight the angle of attack must be that corresponding to zero lift coefficient, supposing the propeller to act directly along the flight path. The equation of motion will then be

$$m\ddot{y} = W + T - D \quad \text{.....................(5)}.$$

This equation may be integrated from a knowledge of T and D as a function of the velocity, but for our present purpose it will be sufficient to deal in some detail only with the case where the aeroplane falls purely under the action of gravity, the propeller being switched off. Putting $T = 0$, $K = \rho A k_d$, where k_d is the drag coefficient in this case corresponding to zero lift, $\alpha = m/K$ and $\beta^2 = W/K$, the equation reduces to

$$\alpha\ddot{y} = \beta^2 - \dot{y}^2.$$

Let $z = \dot{y}$, then
$$\ddot{y} = \frac{dz}{dt} = \frac{dz}{dy}\cdot\frac{dy}{dt} = z\frac{dz}{dy}.$$

Hence
$$az\frac{dz}{dy} = \beta^2 - z^2,$$
$$\therefore \int \frac{\alpha}{2} \frac{d(z^2)}{\beta^2 - z^2} = \int dy.$$

Suppose the initial velocity of drop is V ft./sec. so that $y = 0$ when $z = V$, it is easily found that

$$y = \frac{\alpha}{2}\log\frac{\beta^2 - V^2}{\beta^2 - z^2}$$
$$= \frac{\alpha}{2}\log\frac{\beta^2 - V^2}{\beta^2 - \dot{y}^2},$$

hence
$$\dot{y}^2 = \beta^2 - (\beta^2 - V^2)\,e^{-2y/\alpha}.$$

The velocity therefore tends to the limit β, and after falling a distance y the speed is given by

$$v^2 = \beta^2 - (\beta^2 - V^2)\,e^{-2gy/\beta^2}.$$

The limiting speed corresponding to $y \rightarrow \infty$ is

$$\beta = \sqrt{(W/\rho A k_d)} = S\sqrt{[(k_l)_{\max.}/k_d]} \qquad \text{(6)},$$

where S is the landing speed.

A simple calculation shows that the limiting velocity of a normal machine lies in the neighbourhood of 200 to 250 ft. per second, which it will practically attain after falling a few hundred feet, supposing the initial velocity of fall to be approximately the normal speed of flight. In passing, it may be mentioned that this is not the maximum velocity that may be attained by falling under gravity. This is reached at such an angle of attack as makes $k_l^2 + k_d^2$ a minimum.

The limiting speed is practically unaltered by switching on the propeller, since the latter would then be running at a small value of slip.

§ **6.** *Asymmetric Straight Line Flight.* It has been assumed in the previous discussion that the aeroplane in its steady linear motion was symmetrically situated with regard to a vertical plane through the flight path. It has been shown that such a state of affairs is consistent with equilibrium in flight, but it would be interesting to investigate whether there exists any attitude of the machine which, while violating these conditions of symmetry, yet admits of equilibrium. If such be the case

the further question would then be suggested, whether these two types of equilibrium differ in respect of stability, but such a discussion must be postponed till the next chapter.

That steady asymmetric flight is possible can be seen from a comparatively simple consideration. Suppose the machine moving through the air at a certain angle of yaw. There will be brought into existence in addition to the lift and drag forces a component of the resultant force perpendicular to both of these. In order to neutralise this cross-wind force, the machine would require to be banked. There will thus be three quantities, viz. the angle of pitch, the angle of bank or roll, and the angle of yaw, whose variations are at our disposal to satisfy the conditions of equilibrium.

Resolving the forces on the machine along any three directions three independent equations will be obtained which will suffice to determine these quantities. The elevators, rudder and wing-flaps must be supposed set in the positions required to destroy the turning moments about the three directions.

The state of motion here discussed is typical of certain occurrences in flight technically known as *side-slipping*. When the axis of the machine is not in the same vertical plane as the axis of flight so that the relative wind is not parallel to the plane of symmetry the machine is said to side-slip.

§ 7. *Performance Curves*. Performance curves describing the characteristics of the machine at various speeds are usually plotted on the assumption that the atmosphere is of uniform density. The fact that the air forces and the propeller thrust depend directly on ρ the density of the air indicates clearly that the performance curves will not be complete unless allowance is made for variations in this quantity. The mere fact that at a height of 10,000 ft., a not uncommon flying altitude, the density has already dropped to 0·7 of its value at the earth's surface, is a sufficiently clear indication that alterations in the performance curves due to this cause are of considerable importance.

In fig. 125 are plotted the resistance of the wings (curve 1) and of the body (curve 2) at various speeds at ground level for a given machine. Curve 3 represents the total resistance and 4 the slope of the gliding angle. It should be noticed that since $\tan\theta$, the gliding slope, is given by $-D/L$, where L is of course

constant, the ordinates of curve 4 are practically proportional to those of 3 slightly modified however by the inclusion of propeller slip stream. This is specially marked at high speeds. The product of such ordinates of curve 3 by the corresponding velocity gives the power required to overcome the resistance, also at ground level. This is plotted in fig. 125 in the same diagram as the total horse-power available from the propeller. The difference between the ordinates of these two curves for any speed furnishes a measure of the horse-power available for climb, etc. The upper point of intersection determines the maximum speed of horizontal flight. The difference between the two resistance curves for gliding and for flying is due to the slip stream of the propeller.

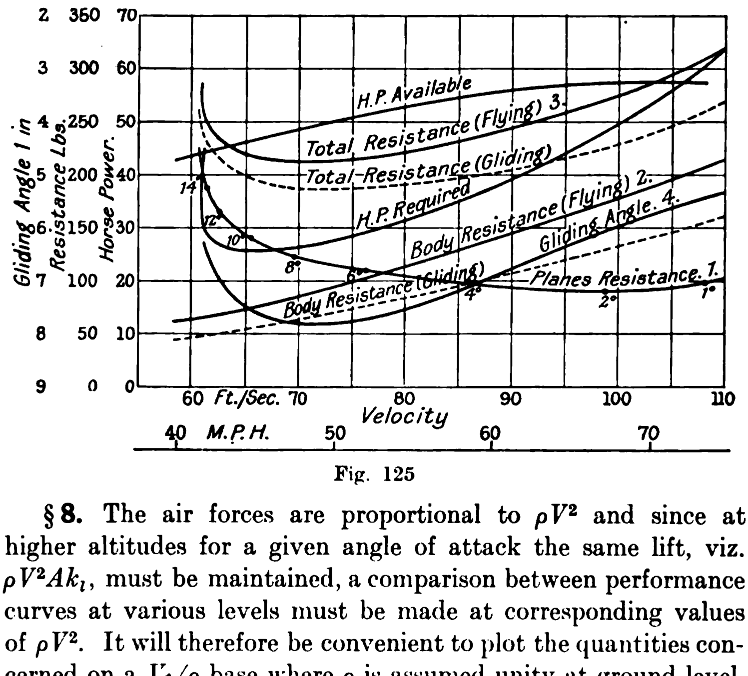


Fig. 125

§ **8.** The air forces are proportional to ρV^2 and since at higher altitudes for a given angle of attack the same lift, viz. $\rho V^2 A k_l$, must be maintained, a comparison between performance curves at various levels must be made at corresponding values of ρV^2. It will therefore be convenient to plot the quantities concerned on a $V\sqrt{\rho}$ base where ρ is assumed unity at ground level. On this base the resistance curve of a machine will not change with altitude. This is not so however with the horse-power curves. In the case of the horse-power required to overcome the drag, the latter quantity was multiplied by the corresponding velocity. To obtain this horse-power curve therefore, on a $V\sqrt{\rho}$ base, each original horse-power must now be divided by $\sqrt{\rho}$ and the ordinates are accordingly all increased.

Since at the higher altitudes the velocity of flight must be increased to maintain the lift, the propeller will of necessity be working at a lower slip, and the thrust will drop unless the revolutions be increased proportionately to give the same thrust and slip. At the same time the energy given to the engine has fallen on account of the smaller mass of air taken up by the cylinder at each stroke for these lower densities, and therefore the horse-power delivered up will fall off for the same revolutions. approximately as the density.

§ 9. In fig. 126 is plotted horse-power available from engine at ground level against engine revolutions (curve A). Curve B representing the engine horse-power at a height where the density is ρ is derived by multiplying each ordinate of A by ρ/ρ_0. The system of curves, C, D, E, etc. giving the propeller torque horse-power against revolutions at various constant speeds, intersect curve B at X, Y, Z, etc., and accordingly fix the torque horse-power, forward velocity and revolutions. From these the slip and efficiency are of course at once obtained and by multiplying the

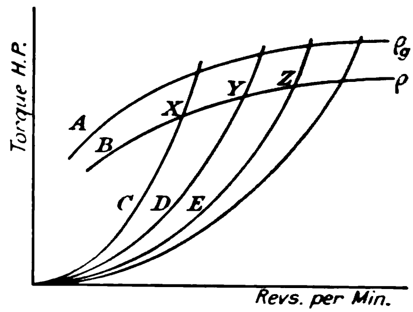


FIG. 126

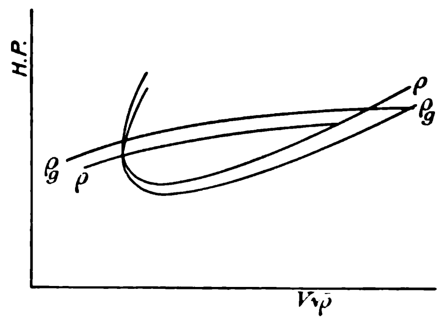


FIG. 127

latter by the torque horse-power the thrust horse-power is derived corresponding to the height. This is now plotted on a $V\sqrt{\rho}$ base in fig. 127. The horse-power required to drive the machine is

immediately calculable and plotted in fig. 127 on a $V\sqrt{\rho}$ base by multiplying the ordinate of the horse-power curve at ground level by $\sqrt{(\rho_g/\rho)}$.

Once more the horse-power available for further climb at each speed and the greatest minimum speed of flight can be derived. It is manifest that as the density diminishes these curves include a constantly decreasing area until ultimately a point is reached at which the maximum and minimum speeds of flight coincide and there is no horse-power available for further climb. This determines the highest altitude to which this machine can ascend. This height is usually termed the *ceiling*. It can be further increased by special treatment of the engine, for example by supplying it with oxygen under pressure, a process usually known as doping the engine.

§ **10.** *Steady Horizontal Circular Flight.* Consider the effect of turning the rudder of a machine in steady horizontal flight. There is immediately brought into play a force on the rudder tending to displace the centre of gravity of the machine laterally. This is quickly nullified, however, by the fact that a rotation about the centre of gravity is set up simultaneously by the moment of this same force about that point, with a consequent cross-wind force produced by the keel surface of the machine.

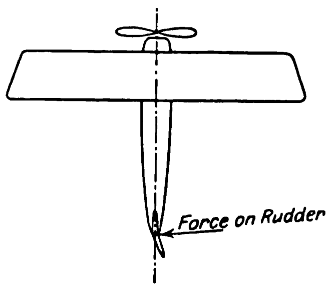


FIG. 128

This cross-wind force supplies the acceleration towards the centre of curvature of the flight path, necessary to maintain the motion. Unless the machine were in this yawed position this force would not come into existence. Fig. 129 illustrates the successive positions adopted by the aeroplane in its flight. At P_1 the machine is flying normally along a straight path and the rudder has just been turned, thus bringing into existence a force on the rudder in the direction as indicated. At P_2, to which position inertia has carried it, the axis is already yawed owing to the moment of this force, but does not lie along the flight path. Sideslipping has occurred in the sense already defined in § 6. In this position moreover the cross-wind force is scarcely greater than

that on the rudder, and accordingly although the machine has rotated about the centre of gravity the path has not yet become perceptibly curved. In the position P_4 the flight path has become curved as a consequence of the increasing cross-wind force, but the machine is still flying yawed and the moment due to the force on the rudder is just balanced by the moment brought into play by the extra force on the outer wing on account of its greater linear velocity. This flight, turning accompanied by side-slipping,

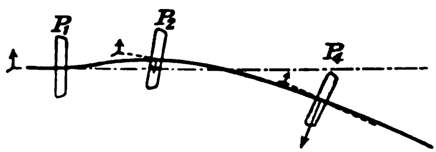


FIG. 129

may be maintained with the machine horizontal by counteracting the increase lift on the outer wing which would involve banking by means of an adjustment of the wing-flaps. It need scarcely be pointed out that the curvature obtained from such a mode of motion is extremely slight and is of no importance from a practical point of view, where rapid turning is usually necessary.

§ **11.** To maintain circular flight a first requirement is evidently the supplying of some external force, of necessity the wind pressure, towards the centre of curvature. In the case already discussed this was obtained by maintaining a continuous yawed flight by the action of the rudder. Other and more convenient means, however, may be adopted to bring the force into existence. If the machine is supposed flying horizontally and then banked, the lift will no longer be vertical as in symmetrical flight, but will have a component perpendicular to the path of the machine and in the direction towards which it is banked. It will be convenient in the first place to discuss the motion of a banked machine with the rudder initially in the symmetrical position and the axis of the machine along

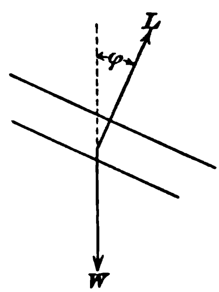


FIG. 130

the flight path. If ϕ be a small angle of bank, W the weight and L the lift, then for horizontal equilibrium of the machine

$$W = L \cos \phi.$$

The horizontal force in the direction of bank equals $L \sin \phi$. This will supply the force required to drive the centre of gravity initially on a circle of radius R where

$$\frac{W}{g} \cdot \frac{V^2}{R} = L \sin \phi.$$

The aeroplane as a body will thus commence to move in a circle, but since no turning moment has been brought into play the axis of the machine will not continue to coincide with the flight path. Side-slipping will therefore immediately take place. As a consequence of the circular motion, the outer wing, moving more rapidly than the inner, experiences a greater lift and drag and two effects immediately become apparent. In the first place the angle of bank will tend to increase, and in the second place this increased drag will set up a turning moment about the centre of gravity augmenting the yaw. The side-slip will be intensified. This will continue until an equilibrium position, if any, is reached determined by the increasing pressure on the tail fin. The tendency towards increased banking may of course be neutralised by an adjustment of the wing-flaps. It is obvious that complete motion in a circle, that is without side-slipping, could not here be obtained without the machine being turned by swinging over the rudder to bring the axis along the flight path.

§ **12.** Generally the above considerations indicate that circular flight may be executed either by the action of the rudder alone or by banking. In both cases, however, side-slipping will result and this effect, as will shortly be rigorously shown, will not be eliminated unless both the rudder and wing-flaps are utilised. Under such circumstances the turning will be most rapid, and the axis of the machine will remain continuously along the flight path, but unlike steady horizontal flight the symmetry will be destroyed. It must be specially emphasised that the main function of the rudder is to turn the axis of the machine, and that of the banking to supply the requisite force towards the centre of the circle.

Suppose the machine has reached the condition of steady horizontal flight with velocity V in a circle of radius R, the rudder

and other controls having been set to the appropriate positions to maintain the steady turning. The angle of bank being ϕ the equations of motion become

$$L \sin \phi = W/g \cdot V^2/R \quad \text{.....................(7)},$$
$$L \cos \phi = W \quad \text{.................................(8)},$$
$$D = T \quad \text{.................................(9)},$$

neglecting the relatively small components of the force on the rudder and the difference in lift and drag on the two wings due to the fact that the outer wing is moving more rapidly than the inner. For a given speed and flight path, the requisite angle of bank is immediately given by

$$\tan \phi = V^2/gR.$$

It follows at once that to fly in a circle of small radius with a given velocity the angle of bank must be steep, but equation (8) indicates at the same time that in order that the machine will not drop the lift must be immediately augmented, demanding an increased angle of attack compared with horizontal flight. This involves a greater expenditure of horse-power.

§ **13.** In general performance curves for any machine usually give a measure of the horse-power required to maintain horizontal flight at various speeds. From such a curve the corresponding horse-power required to maintain circular flight may easily be derived. In either case the horse-power is expended in overcoming the drag. Comparing the two cases, horizontal flight at velocity V and circular flight at the same speed but necessarily at a different angle of attack,

$$H_1 = T_1 V = D_1 V = A\rho\,(k_d)_1\,V^3 \quad \text{...........(10)}$$

for circular flight,

$$H_2 = T_2 V = D_2 V = A\rho\,(k_d)_2\,V^3 \quad \text{...........(11)}$$

for horizontal flight.

Therefore $\qquad H_1 = H_2\,(k_d)_1/(k_d)_2 \quad \text{.....................(12)}.$

Now $\qquad W = A\rho\,(k_l)_1\,V^2 \cos \phi$

from equation (8), and

$$W = A\rho\,(k_l)_2\,V^2$$

for horizontal flight.

Therefore $\qquad (k_l)_1/(k_l)_2 = 1/\cos \phi \quad \text{.....................(13)}.$

Suppose the machine flying in straight line flight with velocity V_1 at the angle of attack corresponding to the lift coefficient $(k_l)_1$, that of the previous circular flight, then

$$W = A\rho\,(k_l)_1\,V_1^2,$$

and

$$W = A\rho\,(k_l)_2\,V^2,$$

as before, where V is the horizontal flight equivalent, from this point of view, to the circular flight.

Therefore $$(k_l)_1/(k_l)_2 = V^2/V_1^2,$$

hence $$V_1 = V\sqrt{\cos\phi} \quad\text{.....................(14).}$$

If H_1' and H_2 be the horse-powers required to drive the machine in horizontal flight at velocities V_1 and V respectively, then

$$H_2 = D_2 V = A\rho\,(k_d)_2\,V^3,$$

and

$$H_1' = D_1 V_1 = A\rho\,(k_d)_1\,V_1^3,$$

therefore

$$(k_d)_1/(k_d)_2 = H_1'/H_2 \,.\, V^3/V_1^3$$
$$= H_1'/H_2 \,.\, 1/(\cos\phi)^{\frac{3}{2}}.$$

This gives finally, using (13),

$$H_1 = H_1'/(\cos\phi)^{\frac{3}{2}} \quad\text{.....................(15),}$$

expressing the horse-power required to maintain the machine in horizontal circular flight, of radius R and velocity V, in terms of the horse-power required to drive the machine at the equivalent velocity of horizontal flight

$$V_1 = V\sqrt{\cos\phi}$$
$$= V/(1 + V^4/g^2R^2)^{\frac{1}{4}} \quad\text{..................(16).}$$

The method of obtaining the horse-power curve for circular flight is at once obvious from these formulae. The horse-power for horizontal flight at velocity $V\sqrt{\cos\phi}$ is divided by $(\cos\phi)^{\frac{3}{2}}$ and plotted on a velocity base at abscissa V.

In fig. 131, AD represents the horse-power available from the propeller of a particular machine, and CB the curve giving the horse-power required at various speeds for horizontal flight. The curves of the type PQ representing the horse-power for circular flight at various radii and speeds are obtained by the method indicated above.

Since SM represents the horse-power used up during the flight, LS measures the surplus horse-power available, say for climb. The curve of the type PSQ, passing through LM, will,

except in the cases about to be mentioned, fix the minimum radius of the circle at the velocity determined by M. If CN is the locus of the points on each curve corresponding to maximum lift coefficient, none of the curves of the system can possibly go beyond CN, and the points of intersection of this system with this line determine the minimum radii of circular flight at the corresponding speeds within the range limited by C and N. The minimum possible radius corresponds to the curve touching AB, in this case approximately 185 ft.

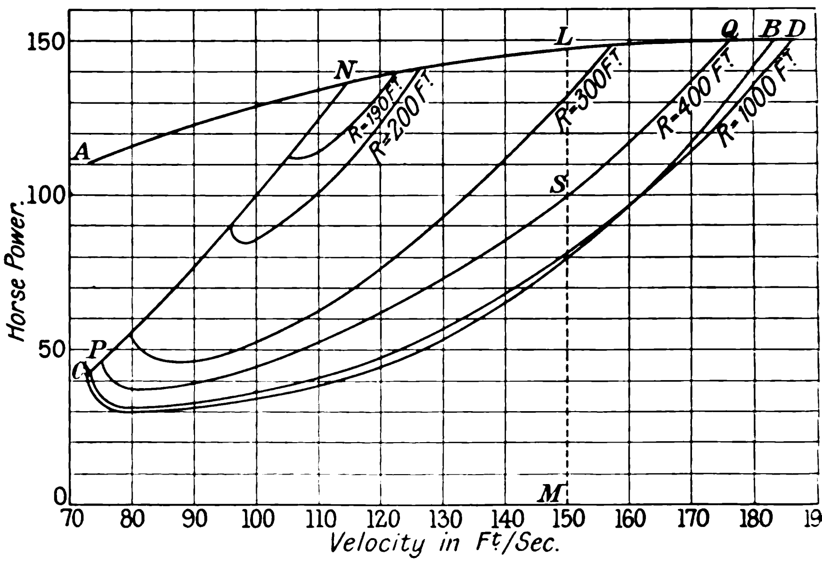


FIG. 131

§ **14.** It must be remembered that in the type of flight contemplated it is necessary that the angle of attack of the machine should correspond to a higher lift coefficient than that of horizontal flight at the same speed, in order to sustain the weight when banked. The corresponding drag coefficient would then be greater or less than that of horizontal flight according to whether the point considered lies above or below the position of minimum drag excluding those cases in which the angle of minimum drag falls between them. It is evidently possible therefore to work at a higher maximum speed in circular than in horizontal flight. This is borne out in fig. 131 by the fact

that the system of "constant radii" curves ultimately cuts *AD* at a series of points, such as *D*, corresponding to a higher speed than that at *B*, viz., the maximum speed of horizontal flight. The difference is however comparatively small, in this case not amounting to more than two miles per hour.

Fig. 132 gives, on the same diagram, the maximum curvature of the path possible, the maximum angle of bank and the angular velocity for various speeds.

A rigorous treatment of the problem of circular flight would be extremely complicated owing to the introduction of certain imperfectly known factors dealing with the variation in wind forces along the wings due to the rotation.

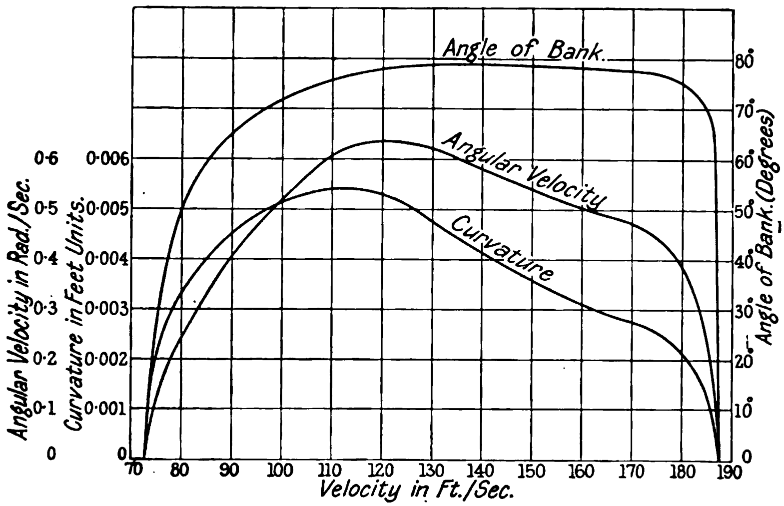


FIG. 132

§ **15.** *Spiral Flight.* It has just been demonstrated that circular flight without side-slipping will take place when both the rudder and banking are made use of. If the machine be supposed at the same time to drop or climb with constant velocity, the path traced out will be a helix. It is proposed to determine the relations that exist between the forces operating, and the various quantities that determine the setting of the machine when the latter is travelling with constant speed V along a vertical helix of inclination θ at each point to the horizontal, formed along the surface of a cylinder of radius r. Assume as before that the controls

have been fixed in the appropriate positions so that the axis of the machine is along the flight path and neglecting the relatively small cross-wind force on the rudder. Defining the angle of bank ϕ by the inclination of the plane of symmetry to the vertical plane through the axis of the machine, and resolving forces along the tangent to the path, along the normal and binormal, fig. 133, the equations of motion become

$$T - D = W \sin\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (17),$$

$$L \cos\phi = W \cos\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (18),$$

$$L \sin\phi = W/g \, . \, V^2/R \quad \ldots\ldots\ldots\ldots\ldots\ldots (19),$$

since the normal is along the radius of the cylinder and the radius of curvature of flight path = $R = r/\cos^2\theta$.

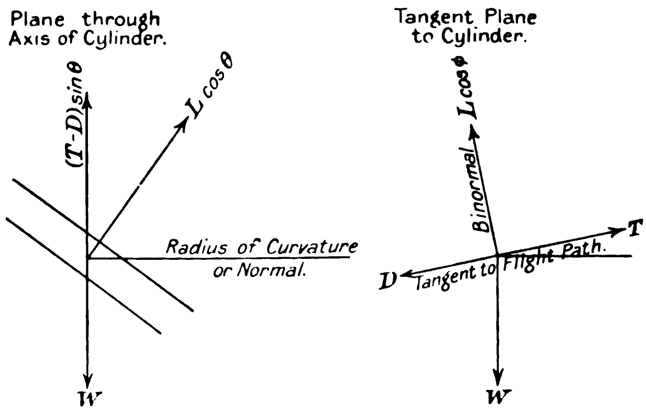


FIG. 133

It follows that the angle of bank for a given helix defined by R and θ and for a given velocity V is determined from

$$\tan\phi = V^2/gR \cos\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots (20).$$

From equation (18) the lift coefficient for this flight becomes

$$k_l = W/\rho A V^2 \, . \, \cos\theta/\cos\phi.$$

Writing equation (17) in the form

$$TV = DV + WV \sin\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots (21),$$

the available horse-power under these conditions may be obtained. The term $WV \sin\theta$ is at once found for various values of V and θ. Following the lines indicated in the case of circular flight, the

horse-power necessary to overcome the drag at various velocities in spiral flight can be derived from the horse-power curve for horizontal flight.

§ **16.** Let $H_1 =$ this horse-power during spiral flight and $H_2 =$ the horse-power for horizontal flight at the same speed, then

$$H_1/H_2 = D_1 V/D_2 V = (k_d)_1/(k_d)_2.$$

Moreover
$$L \cos\phi = W \cos\theta,$$
hence
$$W = A\rho\,(k_l)_1\,V^2 \cos\phi/\cos\theta,$$
and
$$W = A\rho\,(k_l)_2\,V^2$$
for horizontal flight at the same speed.

Therefore
$$(k_l)_1/(k_l)_2 = \cos\theta/\cos\phi.$$

If V_1 be the horizontal velocity corresponding to the same angle of attack as in the spiral flight, then

$$W = A\rho\,(k_l)_1\,V_1^2,$$

therefore
$$V^2/V_1^2 = (k_l)_1/(k_l)_2 = \cos\theta/\cos\phi,$$
so that
$$V_1 = V\sqrt{\cos\phi/\cos\theta} \quad\quad (22).$$

Now
$$H_2 = D_2 V = A\rho\,(k_d)_2\,V^3.$$

Let $H_1' =$ the horse-power required for horizontal flight at the equivalent velocity V_1, then

$$H_1' = A\rho\,(k_d)_1\,V_1^3,$$

therefore

$$(k_d)_1/(k_d)_2 = H_1' V^3/H_2 V_1^3 = H_1'/H_2\,.\,(\cos\theta/\cos\phi)^{\frac{3}{2}}.$$

Hence finally
$$H_1 = H_1'\,(\cos\theta/\cos\phi)^{\frac{3}{2}} \quad\quad (23).$$

Since θ is supposed given and ϕ has been determined in terms of θ, V, and R, this equation provides a means of obtaining the horse-power to overcome the drag during spiral flight at velocity V in terms of the corresponding horse-power for horizontal flight at the velocity $V\sqrt{\cos\phi/\cos\theta}$. This will provide a system of curves for all values of R for each value of θ. By the same process of argument developed for circular flight, the various characteristics, such as the minimum radii of curvature of flight path, are determined. It is evident that the presence of the term $WV\sin\theta$ in equation (21) corresponds to an extra expenditure

of horse-power for climbing beyond that required for circular flight so that the minimum radii of curvature for spiral climbing will be greater than that for circular flight.

It is clear that the problem of spiral flight in its most general form includes as particular cases those of steady circular and straight line flight. The deduction of the results obtained in these two previous cases from the preceding discussion will be left as an exercise to the student and a comparison of the results in these three cases is well worth making. The concluding remarks of § 14 apply with equal force here and it is due to this that the problem of steady vertical spin is not included in the above discussion.

§ 17. *Landing.* Normally when a pilot has arrived above his landing ground he will make a spiral glide to earth, that is he will fly in a spiral without propeller thrust. On approaching close to the ground however, in order to avoid meeting it too sharply he usually flattens out his flight path by quickly switching on his propeller. It occurs not infrequently that omission of this point involves disastrous consequences to the under-carriage. The landing will be most efficiently executed if carried out at a minimum velocity relative to the ground. This is best obtained by landing up-wind at the maximum angle of attack. Immediately on touching, the propeller is switched off, and the speed of the machine reduced by pulling the tail down until the skid drags. It is important to notice that while the machine is in the air its speed is not ultimately increased by switching the propeller on but the flight path is merely flattened out. Such an increase could only occur by an alteration in the elevator setting. If in flattening out the flight path the angle of attack be increased, the propeller not having been switched on, so that the speed of the machine is reduced below that required for sustentation before contact is made with the ground, the aeroplane will simply drop to earth and the bad landing thus produced is known as "pancake landing." Should the angle of attack be increased so quickly that the nose of the machine rises and the forward velocity is practically destroyed, the aeroplane will slide back and smash its tail. Both these types of bad landing are of comparatively common occurrence. Dangerous consequences due to a landing of a pancake type are usually guarded against by a strong under-carriage and by the insertion of shock absorbers.

§ **18.** *Starting.* As with landing, starting is most efficiently executed up-wind, in which position for the same ground speed, the speed relative to the wind is greatest. The elevators are set in the position for minimum drag of the machine, and the propeller having been switched on the aeroplane starts off along the ground. When the speed has passed the landing speed the elevators are set for flight at a higher angle of attack, the machine takes the air, and rises.

§ **19.** The preceding discussions have been confined almost entirely to questions that arise when the aeroplane is moving into air at rest, or those that can be reduced to this, as for example flight in a steady wind. A complete treatment of the more general problem of the behaviour of a particular machine under variable conditions such as the effect upon its motion of a series of gusts is a subject of considerable complexity. One particular class of cases however is of extreme practical importance. Suppose the machine moving with steady motion in a particular path be subjected to a slight disturbance, the question naturally arises whether the aeroplane will return ultimately to the same conditions of flight, and whether it is possible so to design the machine that this desirable result will be achieved. This is concerned with the problem of stability.

CHAPTER XII

STABILITY—MATHEMATICAL THEORY

§ 1. *General Discussion.* The problem of stability treats generally with the effect of a disturbing impulse upon any mechanical system, and more particularly with the state of motion that results from it. For convenience the subject may be divided into two parts. Under the first heading there fall to be considered those cases in which the mechanical system existed previous to the disturbance in a state of static equilibrium. This might be typified by the case of a pendulum, initially as rest, and upon which is impressed a slight impulse. Statical stability, as here illustrated, will concern itself with the future motion of the pendulum and decides whether the original condition of equilibrium will ultimately be reinstated. Aeronautics dealing usually with machines in flight and not at rest, this type of stability will be ignored. In the second place a state of motion under the existing forces having been proved possible, at some point in its history a slight disturbing force is impressed on the system. Dynamical stability deals with the resulting motion and concerns itself with the question as to whether on the removal of the disturbing force the original state of motion will be eventually resumed. The meaning of the problem and the method of attack is illustrated by the following example from the dynamics of a particle.

§ 2. Suppose a particle of mass m sliding from rest under gravity down the lowest generator of a smooth cylindrical tube of circular section filled with fluid inclined at an angle α to the horizontal. It is required to determine its subsequent motion and to find the effect of a slight displacement from this state of motion, assuming the fluid resistance proportional to the velocity at any time.

The forces acting upon the particle are the weight mg, the resistance Kmv against the direction of motion and the reaction

R along the radius of the tube. The resulting acceleration down the generator will be dv/dt, and hence

$$m\frac{dv}{dt} = mg\sin\alpha - Kmv \quad \text{..................(1)},$$

$$\therefore \; dv/(g\sin\alpha - kv) = dt,$$

and the velocity is given by

$$v = (g/K)\sin\alpha\,.\,(1 - e^{-Kt}),$$

since $v = 0$ where $t = 0$. Since after the lapse of some time e^{-Kt} becomes negligible, the velocity reaches the terminal value $g\sin\alpha/K$.

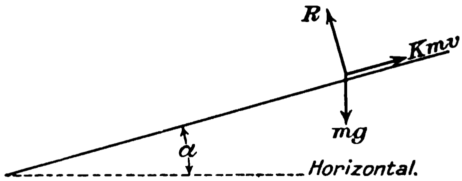


FIG. 134

Equation (1) contains a full statement of the history of the particle. Suppose after a time t, it receives a small impulse giving it an additional velocity u perpendicular to its path, so that at time t it occupies the position P and has a component of velocity $r\dot{\theta}$ along the circumference of the cross-section (fig. 135). The forces acting on the particle along the tangent are now $Kmr\dot{\theta}$ due to the fluid, and $mg\sin\alpha\sin\theta$ due to the weight. Writing the component acceleration along the tangent as $r\ddot{\theta}$ the equation of the component motion round the circle becomes

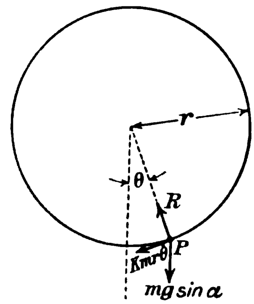


FIG. 135

$$mr\ddot{\theta} = -Kmr\dot{\theta} - mg\sin\alpha\sin\theta.$$

Assuming the displacement θ is small this takes the form

$$\ddot{\theta} + K\dot{\theta} + \frac{g\sin\alpha}{r}\theta = 0.$$

The solution of this differential equation is

$$\theta = Ae^{\lambda_1 t} + Be^{\lambda_2 t} \quad \text{.....................(2)},$$

where λ_1 and λ_2 are the two roots of the equation

$$\lambda^2 + K\lambda + g \sin \alpha/r = 0,$$

$$\text{i.e., } \lambda = -K/2 \pm \sqrt{(K^2/4 - g \sin \alpha/r)}.$$

Four cases fall to be considered.

(*a*) $K^2/4 > g \sin \alpha/r$. This involves that the two values for λ are real and negative, so that the small angular displacement from the lowest generator gradually diminishes to zero according to equation (2) and the motion tends to become the same as that before the displacement. The motion of the particle in this case, where it tends and finally does return to its previous motion, is an illustration of a certain form of stability.

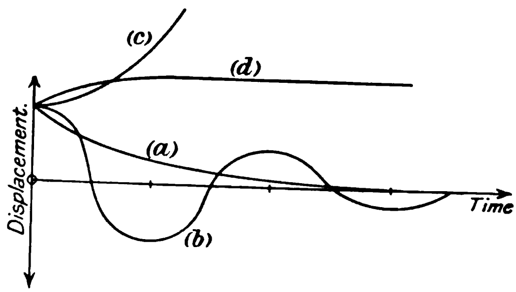


FIG. 136

(*b*) $K^2/4 < g \sin \alpha/r$. The quantity determined by the root sign is now purely imaginary, and the expression for θ takes the form

$$\theta = e^{-Kt/2}(Ae^{qit} + Be^{-qit})$$
$$= e^{-Kt/2}(A_1 \cos qt + B_1 \sin qt),$$

where $$q^2 = K^2/4 - g \sin \alpha/r.$$

The angular displacement of the particle now oscillates about the lowest generator in constantly decreasing amplitudes, until finally it dies out. This likewise illustrates a particular type of stability.

The variation in θ with time is shown in fig. 136 for the two cases.

(*c*) From these formulae may be derived those applicable to the case where the particle, rolling along the top generator on the outside of the tube, is given a similar slight displacement.

These may be obtained either directly or by simply changing the sign of r in the above equations so that

$$\lambda = -K/2 \pm \sqrt{(K^2/4 + g \sin \alpha / r)}.$$

It is obvious that there is always one value for λ which is positive, and accordingly the expression for θ involves the term $e^{\lambda_1 t}$ where λ_1 is positive. θ therefore increases with time, and the path of the particle tends continually to diverge from the highest generator; the original motion will never be resumed. This is an illustration of instability, and the continuous divergence from the path of equilibrium is represented in fig. 136.

(*d*) When the particle is in motion down a line of greatest slope of an inclined plane of infinite extent, the equations of motion can be similarly derived, either as a limiting case of the above or directly. If as before the particle be given a small impulse perpendicular to its path so that at any time it is displaced a distance x from the original line of motion, then its acceleration in that direction is given by

$$m\ddot{x} = \text{retardation}$$

$$= -mk\dot{x}.$$

Hence $\qquad x = A + Be^{-Kt}$ and $\dot{x} = -KBe^{-Kt}$.

As time goes on the particle tends to return to its original motion, since $\dot{x}$ dies out, but moves ultimately along a line parallel to, but displaced a distance A from, the original path. As far as the actual path is concerned the stability is of the neutral type, but, since the speed is ultimately of the same magnitude, and in the same direction as that which would have existed without the disturbance, as far as the velocity is concerned this is a case of stability.

§ **3.** Before leaving this example and approaching the question of stability from a more general point of view it must be remarked that the criterion for stability here enunciated amounts to the statement that only when the values of λ in the exponential terms obtained for the displacement at any time are either always negative or, if complex, have the real part negative, the motion is stable. In this statement are not included such cases as where the stability is neutral or oscillatory as would arise when no resisting force is present.

§ 4. In seeking for a sufficiently inclusive definition and an adequate criterion for stability, the effects of every possible case of disturbance upon various types of motion must be considered. An analysis of the nature of the disturbances leads immediately to a separation under two heads. In the first place the mechanical system, be it a particle or such a complicated mechanism as an aeroplane or even such a medium as a fluid, may be supposed to undergo a displacement of infinitely small amount. The question for investigation is whether the resulting motion will tend to approach or diverge from that previously existing. It must be pointed out that if the original equations of motion are continuous functions of position, velocity, etc., an infinitely small displacement, even in the case where the motion tends to diverge, will never attain a finite value in a finite time. Thus, suppose a thin rod balanced vertically on an end is subjected to a displacement of infinitely small amount from that position of unstable equilibrium, then the moment brought into play tending to increase the displacement is infinitely small, increasing at a finite rate and therefore never becomes finite. At the same time the mere fact that the displacement increases in magnitude is normally taken as an indication that the position is one of instability, the implication being that if a small finite disturbance were impressed upon it, a similar continuous increase in displacement would follow.

This consideration immediately suggests a second classification of stability in relation to disturbance. It is conceivable, and numerous illustrations are easily found, that the mechanical system might be stable in this sense to displacements of infinitely small amount or even of finite amount, but unstable to those exceeding a certain magnitude. It would be necessary, in such an event, to state accurately the nature of the disturbance, as for example, that the system was stable to a particular displacement of amount δ. As far as the application of any investigation of this problem to practice is concerned, as in the case of an aeroplane in flight, it is evident that it is really the second view of this question which approaches most nearly to actual conditions. On the other hand it can be contended that the real effect of a disturbance of relatively large amount is to throw the mechanism to a new path:

(*a*) whose ultimate divergence from the original path must be determined, and

(*b*) whose stability requires investigation from the point of view of later slight disturbances.

Generally a complete investigation of the problem of stability demands three separate steps:

(1) A determination of the divergence of the system when subjected to an exceedingly small disturbance.

(2) A determination of the new path traced out by the system, when subjected to a relatively large disturbance, to discover whether it ultimately converges to the original path.

(3) An investigation of the effect on this path of an exceedingly small disturbance, to determine how far this affects the conclusions arrived at in (2).

The present discussion limits itself to an investigation of the stability, or otherwise, of the motion in certain types of equilibrium flight of an aeroplane when subjected to disturbances of such small magnitude that second order quantities may be neglected.

§ 5. *Stability of the Aeroplane in Straight Line Flight.* In tracing out the path of a body in space the general convention is to express the motion, etc., with reference to a system of rectangular axes, presumed fixed in space. As far as the actual path is concerned this is the most instructive means of description, but if the body be other than a particle, an aeroplane for example, certain factors come into operation tending to destroy the simplicity of the method. A rigid body in general is capable of motion involving six degrees of freedom of which three are translational and the remainder rotational. A complete discussion of the motion will involve equations containing the moments and products of inertia of the body at the instant under consideration about the three fixed axes of reference, and these inertia terms will accordingly vary with time. The problem of the stability of an aeroplane can be discussed purely with reference to such fixed axes, but it is evident that, for any type of motion other than straight linear, it will be necessary in the formation of the equations to resolve the moments of inertia of the aeroplane referred to three axes fixed in the body about the axes fixed in space. As pointed out, on account of the complications arising from the fact that this resolution must vary from point to point of the flight path, the method of fixed axes becomes extremely inconvenient, and some other system of axes must be selected to

obviate this difficulty before much progress is to be expected in the development of the subject.

To Prof. Bryan is due the credit for providing a method of discussion of this problem, which, apart from the initial difficulties, is at once simple, effective and suggestive*. As pioneer in the subject he has set out the equations of motion for small oscillations, and given the first impetus to the development from a mathematical standpoint. In the hand of Bairstow†, they have been retransformed and interpreted in such a manner as to make them immediately applicable to problems of performance and design of aeroplanes.

§ 6. *Moving Axes.* To overcome the difficulties inherent in the method of referring the motion continuously to the same set of axes fixed in space, the problem will be attacked by a method due originally to Euler, known as "The method of Moving Axes." Imagine a set of rectangular axes fixed in the body, a part of it so to speak. The motion of the body will be completely known when the exact manner of motion of these three axes is known with respect to any set of axes fixed in space. Let this latter set be so chosen as to coincide

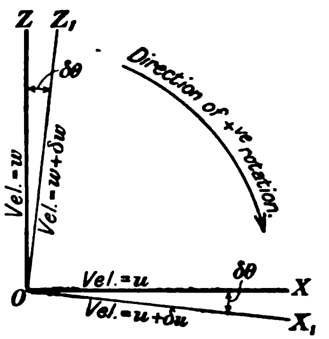


FIG. 137

at the instant under consideration with the three moving body axes. Euler's method is virtually a discussion of the motion of the body with respect to this convenient system of axes fixed in space and left behind the body in its motion. Thus from instant to instant a different set of fixed axes are continually being selected as axes of reference and then discarded. As illustration, consider the case of two dimensions. Let OX and OY, fig. 137, be two axes fixed in the body occupying the positions OX, OY and OX_1, and OY_1 at two successive instants. Let u and w be the component velocities of the body along the fixed axes which coincide with OX and OY at the instant, and $u + \delta u$ and $w + \delta w$ the corresponding components for the consecutive

* *Stability in Aviation*, by Prof. Bryan.

† Advisory Committee Reports.

set of fixed axes. It is required, in the first place, to find an expression for the acceleration of the body when in the position determined by OX, OY. If $\delta\theta$ be the angle between the two sets of fixed axes, then the components of $u + \delta u$ and $w + \delta w$ along the direction OX occurring, as they do, after an interval of time δt are $(u + \delta u)\cos\delta\theta$ and $(w + \delta w)\sin\delta\theta$. The increment in velocity therefore along the fixed direction OX is

$$\begin{aligned}(u + \delta u)&\cos\delta\theta + (w + \delta w)\sin\delta\theta - u\\ &= u + \delta u + w\,\delta\theta - u\\ &= \delta u + w\,\delta\theta \text{ to the first order of small quantities.}\end{aligned}$$

Hence the acceleration of the body along the fixed direction OX, when its moving axis OX happens to coincide with this fixed direction,

$$= \underset{\delta t \to 0}{\mathrm{L}} \frac{\delta u + w\delta\theta}{\delta t} = \frac{du}{dt} + w\frac{d\theta}{dt} \qquad\text{.................(3).}$$

Similarly the acceleration along the fixed direction OY momentarily coincident with the corresponding axis in the body is

$$\frac{dw}{dt} - u\frac{d\theta}{dt} \qquad\text{...........................(4).}$$

Without labouring the point it may be noted that du/dt, under such a system, is thus not the acceleration but simply the limiting ratio to the increment in time, of the velocity as measured along the two sets of different axes.

§ **7.** The more general case applicable to three dimensions can be treated in precisely the same manner. If u, v and w be the component velocities, defined as above, along the directions OX, OY and OZ respectively, then by resolving the new velocities $u + \delta u$, $v + \delta v$, $w + \delta w$ measured along the system of axes coincident after an interval of time δt, and proceeding as in the simpler case discussed, the following equations are obtained, positive rotation being X to Y, Y to Z and Z to X.

$$\text{Acceleration in direction } OX = X = \frac{du}{dt} - v\dot{\theta}_3 + w\dot{\theta}_2 \quad\text{......(5),}$$

$$\text{,,} \qquad \text{,,} \qquad \text{,,} \qquad OY = Y = \frac{dv}{dt} - w\dot{\theta}_1 + u\dot{\theta}_3 \quad\text{......(6),}$$

$$\text{,,} \qquad \text{,,} \qquad \text{,,} \qquad OZ = Z = \frac{dw}{dt} - u\dot{\theta}_2 + v\dot{\theta}_1 \quad\text{......(7).}$$

where $\dot{\theta}_1$, $\dot{\theta}_2$, $\dot{\theta}_3$ are the angular velocities of the body at the instant under consideration about the three axes OX, OY, OZ fixed in space and coincident with the corresponding moving axes of the body. For a complete discussion of this method of dynamical analysis the student is referred to Routh's *Advanced Rigid Dynamics*, Chapter I.

The fact that different systems of new axes are continually being adopted and discarded tends to destroy the utility of this method for tracing out the flight path, but at the same time it provides a simple means of obtaining a clear idea of the nature of the motion at any given instant. It is proposed to apply these equations to the case of an aeroplane in flight and to determine whether, when subjected to certain disturbing forces, its motion tends to converge to, or diverge from, that previously existing.

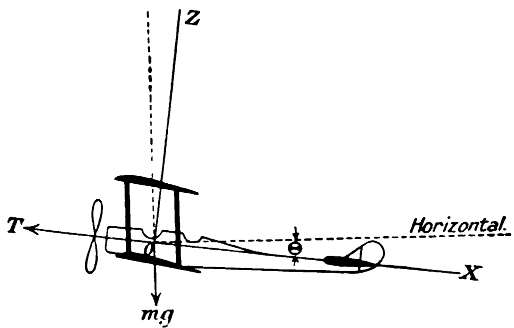


FIG. 138

§ **8.** *Longitudinal Disturbances.* The aeroplane, supposed moving steadily in symmetrical straight line flight, is given a small impulse in any direction in the plane of symmetry. This instantaneously has the effect of changing the velocity of flight and consequently also the resistance of the various parts by corresponding small amounts. Since the motion and the disturbance are both symmetrical, the problem can be treated in two dimensions, i.e., in the plane of symmetry. The axes to be chosen are those coincident at the moment with axes fixed in the body; these latter prior to the disturbance were along, and perpendicular to, the flight path, the origin being at the centre of gravity. The positive directions are so selected as to represent the forces of lift and drag by positive sign. The angle of pitch

Θ is supposed positive when its direction corresponds to an increase of angle of attack. In such a case the forward velocity of the machine to give rise to such positive forces must be negative. Positive pitching moment is measured in the same direction as Θ.

If mX and mZ be the components of the wind forces in the directions OX and OZ at some time after the disturbance, then the component forces producing accelerations along these directions are:

$$OX \ldots\ldots \quad mg \sin \Theta + mX - T,$$
$$OZ \ldots\ldots - mg \cos \Theta + mZ,$$

where T is the propeller thrust assumed acting along OX.

Using equations (3) and (4) for the motion of the centre of gravity, the equations of motion become:

$$m\left(\frac{dU}{dt} + W\frac{d\Theta}{dt}\right) = mg \sin \Theta + mX - T \quad \ldots\ldots(8),$$

and $$m\left(\frac{dW}{dt} - U\frac{d\Theta}{dt}\right) = - mg \cos \Theta + mZ \ldots\ldots\ldots\ldots(9),$$

where U and W are the velocities in the space directions OX and OZ.

For the rotational motion about the centre of gravity we have

$$mK^2 \frac{d^2\Theta}{dt^2} = mK^2 . M \ldots\ldots\ldots\ldots\ldots\ldots\ldots(10),$$

where $mK^2 . M$ is the pitching moment due to the air forces, neglecting gyroscopic effects of engine and propeller, and supposing the propeller thrust passes through the centre of gravity.

The three equations given above are perfectly general and therefore may be applied to the aeroplane immediately before, and some time after, the disturbance. Let U_0, W_0, X_0, Z_0, T_0, Θ_0 and M_0 be the values for the velocities, force coefficients, etc., during steady flight, and let these become $U_0 + u$, $W_0 + w$, $X_0 + X$, $Z_0 + Z$, $T_0 + T$, $\Theta_0 + \theta$ and $M_0 + M$ after the impulse where u, w, etc. are so small that second order terms may be neglected. The following two sets of three equations are at once given from (8), (9) and (10)*,

* It should be noted that $\frac{d\Theta_0}{dt}$ is intended to represent what is more commonly written as $\left(\frac{d\Theta}{dt}\right)_0$, *i.e.* the value of $\frac{d\Theta}{dt}$ at $t = 0$.

$$m\left(\frac{dU_0}{dt}+W_0\frac{d\Theta_0}{dt}\right)=mg\sin\Theta_0+mX_0-T_0 \quad \text{...(11)},$$

$$m\left(\frac{dW_0}{dt}-U_0\frac{d\Theta_0}{dt}\right)=-mg\cos\Theta_0+mZ_0 \quad \text{.........(12)},$$

$$mK^2\frac{d^2\Theta_0}{dt^2}=mK^2M_0 \quad \text{........................(13)},$$

and

$$m\left(\frac{dU_0}{dt}+W_0\frac{d\Theta_0}{dt}+\frac{du}{dt}+w\frac{d\Theta_0}{dt}+W_0\frac{d\theta}{dt}\right)$$

$$=mg\sin\Theta_0+mX_0-T_0+mg\theta\cos\Theta_0+mX-T \quad \text{...(14)},$$

$$m\left(\frac{dW_0}{dt}-U_0\frac{d\Theta_0}{dt}+\frac{dw}{dt}-u\frac{d\Theta_0}{dt}-U_0\frac{d\theta}{dt}\right)$$

$$=-mg\cos\Theta_0+mZ_0+mg\theta\sin\Theta_0+mZ \quad \text{...(15)},$$

$$mK^2\frac{d^2\Theta_0}{dt^2}+mK^2\frac{d^2\theta}{dt^2}=mK^2M_0+mK^2M \quad \text{......(16)}.$$

These six equations apply with perfect generality to the case where the motion of the aeroplane may be considered purely in the symmetrical plane, without any limitation as regards steadiness of flight. In the present case where the axis of X is taken as initially tangential to the flight path, $W_0=0$, and this does not of course impose any limitation on the nature of the flight but is merely the result of a convenient choice of axes. For steady flight, none of the quantities U_0, W_0 and Θ_0 vary with time, and therefore the left-hand sides of the first three equations vanish. Steadiness therefore demands

$$mg\sin\Theta_0+mX_0-T_0=0 \quad \text{..................(17)},$$

$$-mg\cos\Theta_0+mZ_0=0 \quad \text{..................(18)},$$

$$M_0=0 \quad \text{..................(19)}.$$

The second set of equations, modified according to these deductions, applicable immediately subsequent to the disturbance then reduce to

$$m\frac{du}{dt}=mg\theta\cos\Theta_0+mX-T \quad \text{.........(20)},$$

$$m\left(\frac{dw}{dt}-U_0\frac{d\theta}{dt}\right)=mg\theta\sin\Theta_0+mZ \quad \text{..............(21)},$$

$$mK^2d^2\theta/dt^2=mK^2M \quad \text{..............................(22)}.$$

§ **9.** The latter three are the equations for small disturbances, and upon them attention will be focused for the present. Each of the quantities X, Z, T and M are actually determined by the shape of the machine and the small increments in velocities, etc., and as such must be considered as functions of the variables u, w and $\dot{\theta}$. In effect, any variation in velocity, accompanied as it would be by a change in angle of attack, gives rise to increments in these air forces, etc. Accordingly since mX represents this increase in force, using Taylor's expansion, it follows that

$$mX = m\left(u\frac{\partial X}{\partial U} + w\frac{\partial X}{\partial W} + \dot{\theta}\frac{\partial X}{\partial \dot{\Theta}}\right) \quad \text{...............(23)},$$

$$mZ = m\left(u\frac{\partial Z}{\partial U} + w\frac{\partial Z}{\partial W} + \dot{\theta}\frac{\partial Z}{\partial \dot{\Theta}}\right) \quad \text{...............(24)},$$

$$mK^2M = mK^2\left(u\frac{\partial M}{\partial U} + w\frac{\partial M}{\partial W} + \dot{\theta}\frac{\partial M}{\partial \dot{\Theta}}\right) \quad \text{.........(25)},$$

$$T = u\frac{\partial T}{\partial U} + w\frac{\partial T}{\partial W} + \dot{\theta}\frac{\partial T}{\partial \dot{\Theta}} \quad \text{.....................(26)},$$

neglecting as before terms of higher order than the first. It must be remarked that θ modifies the wind forces only in so far as it affects u and w, and therefore cannot appear as an independent variable. Each of the coefficients of u, w and $\dot{\theta}$ in the above four equations has a special significance, and it is proposed to consider these individually and interpret their bearing on the question at issue. It must in the first place be emphasised that a vital distinction exists between the longitudinal force mX and the drag, and between the normal force mZ and the lift. For this and later purposes it is necessary to find the exact expression for mX and mZ. Prior to the disturbance these forces respectively coincide in direction, but after the application of the impulse the one system is inclined to the other by a small angle α representing the change in angle of attack in the plane of symmetry. The relative wind, U', has a component $U' \sin \alpha$ along the Z-axis. The velocity of the machine itself along this axis is w and therefore $U' \sin \alpha = -w$, or since α is small $\quad \alpha = -w/U' = w/U,$

U' being practically equal to $-U$. The relation that holds between these forces thus takes the form

$$mX = D\cos\alpha - L\sin\alpha = D - L\alpha \quad \text{approx.},$$
$$mZ = D\sin\alpha + L\cos\alpha = D\alpha + L \quad \text{approx.},$$

where as before L and D are the lift and drag forces, and the angle α is supposed small. Finally

$$mX = D - Lw/U \text{ .... (27)},$$

$$mZ = Dw/U + L \text{ .... (28)}.$$

In particular $mX_0 = D_0$ and $mZ_0 = L_0$.

These equations will now be utilised for the derivation of the force derivatives.

§ **10.** (i) $m\dfrac{\partial X}{\partial U}$ or, as more commonly written, mX_u represents the rate of variation of the longitudinal force with forward velocity alone, the other variables being supposed constant. Experiment shows that the resisting forces are practically

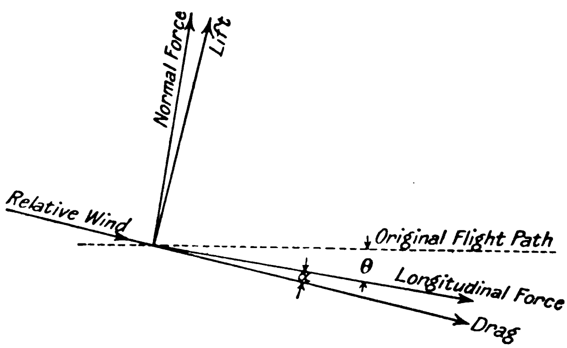


FIG. 139

proportional to the square of the velocity and for an aeroplane independent of the acceleration, hence

$X = KU^2$ where U is the velocity of the machine along the X-axis as before and is essentially negative.

$$\therefore\ \partial X/\partial U = 2KU = 2U \times X/U^2 = 2X/U.$$

At the moment under discussion this velocity is $U_0 = -U_0'$ where U_0' is the positive speed of the wind, and therefore X_u becomes
$$\left(\frac{\partial X}{\partial U}\right)_0 = 2\frac{X_0}{U_0} = \frac{2}{U_0}\cdot\frac{D_0}{m} = -\frac{2}{U_0'}\cdot\frac{D_0}{m}.$$

This quantity is thus easily determined by experiment.

(ii) $m\dfrac{\partial X}{\partial W}$ in the same way represents the rate of variation

of the longitudinal force with the velocity W along the axis of Z, the other quantities remaining constant. w decreases the angle of attack by an amount w/U, i.e. $\alpha = -w/U$, and therefore this derivative represents the slope at $\alpha = 0$ of the normal force curve, already referred to. Differentiating equation (27) it follows that mX_w is likewise given by

$$\frac{1}{U_0'}\cdot\left[L_0 - \left(\frac{dD}{d\alpha}\right)_0\right].$$

(iii) $m\dfrac{\partial X}{\partial\dot{\Theta}}$ or as it will be written in future mX_q. No simple expression can be found for this quantity, which must therefore depend for its determination on a special experiment. Its significance however can easily be understood from a simple consideration. Owing to the velocity of pitch the tail of the machine moving downwards will meet the air at an increased angle of attack, and is therefore equivalent to a variation in the forces on the tail and therefore on the machine, the variation in the X direction being measured by this quantity.

Following the same type of argument it will easily be seen that

(iv) $m\dfrac{\partial Z}{\partial U} = mZ_u = 2\dfrac{Z_0}{U_0} = 2\dfrac{L_0}{U_0} = -2\dfrac{L_0}{U_0'}$.

(v) $m\dfrac{\partial Z}{\partial W} = mZ_w =$ slope of normal force curve with α at $\alpha = 0$

$$= -\frac{1}{U_0'}\left[\left(\frac{dL}{d\alpha}\right)_0 + D_0\right].$$

(vi) $m\dfrac{\partial Z}{\partial\dot{\Theta}} = mZ_q$ has a similar interpretation to (iii).

(vii) $m\dfrac{\partial M}{\partial U} = mM_u = 2\dfrac{M_0}{U_0} = 0$ for steady flight.

(viii) $m\dfrac{\partial M}{\partial W} = mM_w = -\dfrac{dM}{d\alpha}\times\dfrac{1}{U_0'}$.

(ix) $m\dfrac{\partial M}{\partial\dot{\Theta}} = mM_q$ has a similar interpretation to (iii).

With the exception of $\partial T/\partial U$, which may be included in $\partial X/\partial U$, the remaining derivatives are comparatively unimportant and will for the present be neglected. If necessary corrections for them can be applied.

§ 11. Substituting in the three equations (20), (21), (22), these approximate values X_u, etc., for X, etc. derived in (23), (24), (25) and (26), they reduce to

$$\frac{du}{dt} = (g \cos \Theta_0)\,\theta + uX_u + wX_w + \frac{d\theta}{dt} X_q \quad \text{...(29 a)},$$

$$\frac{dw}{dt} - U_0 \frac{d\theta}{dt} = (g \sin \Theta_0)\,\theta + uZ_u + wZ_w + \frac{d\theta}{dt} Z_q \quad \text{......(29 b)},$$

$$\frac{d^2\theta}{dt^2} = wM_w + \frac{d\theta}{dt} M_q \quad \text{..............................(29 c)}.$$

Writing the operator d/dt as D so that $du/dt = DU$, $d^2\theta/dt^2 = D^2\theta$, etc., these equations take the symbolic form

$$(D - X_u)\,u - X_w w + (-X_q D - g \cos \Theta_0)\,\theta = 0 \quad \text{...(30 a)},$$

$$-Z_u u + (D - Z_w)\,w - [g \sin \Theta_0 + (U_0 + Z_q)\,D]\,\theta = 0 \quad \text{...(30 b)},$$

$$-M_w w + (D^2 - M_q D)\,\theta = 0 \quad \text{...(30 c)}.$$

A complete solution of these equations will determine u, w and θ as functions of t. This will be obtained by eliminating any two of these quantities between the above three equations by the following simple step by step process. By multiplying (30 a) by Z_w and operating on (30 b) by $(D - X_u)$ and subtracting, an equation is obtained involving only w, θ and their derivatives with regard to time. Using this with equation (30 c) in a similar manner a single relation involving θ and its derivatives will be obtained. This method evidently is equivalent to the process of eliminating by the ordinary algebraic method the three quantities u, w and θ between the three relations (30 a), (30 b) and (30 c), and no matter which of these three is ultimately left, there will be the same symbolic relation operating upon it, viz.,

$$\begin{vmatrix} D - X_u, & -X_w, & -X_q D - g \cos \Theta_0 \\ -Z_u, & D - Z_w, & -g \sin \Theta_0 - (U_0 + Z_q)\,D \\ 0, & -M_w, & D^2 - M_q D \end{vmatrix} (u, w, \theta) = 0 \quad \text{...(31)}.$$

It follows that the solution of this equation of the fourth order of differential coefficients must therefore be such, that u, w and θ will be expressible in exactly similar form as far as t is concerned.

Assuming that any of these quantities can be written pro-

portional to $e^{\lambda t}$ as in the case of the motion of the particle in the tube already discussed, then

$$De^{\lambda t} = \lambda e^{\lambda t}, \quad D^2 e^{\lambda t} = \lambda^2 e^{\lambda t}, \text{ etc.,}$$

and λ may be substituted for D.

Hence from (31)

$$\begin{vmatrix} \lambda - X_u, & -X_w, & -X_q\lambda - g\cos\Theta_0 \\ -Z_u, & \lambda - Z_w, & -g\sin\Theta_0 - (U_0 + Z_q)\lambda \\ 0, & -M_w, & \lambda^2 - M_q\lambda \end{vmatrix} = 0 \quad \text{...(32).}$$

Expanding the determinant by the ordinary method this takes the form

$$A\lambda^4 + B\lambda^3 + C\lambda^2 + D\lambda + E = 0 \quad \text{..............(33),}$$

where $A = 1$,

$$B = -(M_q + X_u + Z_w),$$

$$C = \begin{vmatrix} Z_w, & U_0 + Z_q \\ M_w, & M_q \end{vmatrix} + M_q X_u + \begin{vmatrix} X_u, & X_w \\ Z_u, & Z_w \end{vmatrix},$$

$$D = -\begin{vmatrix} X_u, & X_w, & X_q \\ Z_u, & Z_w, & U_0 + Z_q \\ 0, & M_w, & M_q \end{vmatrix} - gM_w \sin\Theta_0,$$

$$E = gM_w \begin{vmatrix} X_u, & \cos\Theta_0 \\ Z_u, & \sin\Theta_0 \end{vmatrix}.$$

§ **12.** All the coefficients in this equation can be determined for a given machine from a series of experiments on a model, and an examination of the four roots of this biquadratic will indicate whether or not the machine so constructed is stable. It is important to investigate the significance of each root. The differential equation (31) being of the fourth order will have as its complete solution an expression of the form, say

$$\theta = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t} + A_3 e^{\lambda_3 t} + A_4 e^{\lambda_4 t} \quad \text{.........(34),}$$

where A_1, A_2, etc. are four constants depending for their values on the initial conditions of displacement, and where $e^{\lambda_1 t}$, $e^{\lambda_2 t}$, etc. are four independent particular solutions of (33). For convenience in solution, it was assumed that θ was proportional to $e^{\lambda t}$. The four roots of equation (33) will give four values of λ, say $\lambda_1, \lambda_2, \lambda_3, \lambda_4$, each corresponding to a particular solution of the differential equation, and under these conditions the above expression for θ will give the general solution required.

Referring back to the general discussion on stability at the beginning of this chapter, it is evident in the first place that the conditions there demanded will be violated if any one of the four roots be real and positive. On the other hand, if these four quantities be real and negative, the motion will be stable, for θ will then ultimately die out. There remains only to be discussed the case of complex roots. If $\lambda_1 = \alpha + i\beta$, then

$$e^{\lambda_1 t} = e^{(\alpha+i\beta)t} = e^{\alpha t}(\cos\beta t + i\sin\beta t).$$

This term will consequently oscillate in value as time proceeds with a period $2\pi/\beta$ and amplitude $e^{\alpha t}$. If α is positive, the magnitude of the oscillations increases indefinitely with time and the motion, as far as this term is concerned, must be considered unstable. If α is zero so that the root is purely imaginary, the motion is simply harmonic and is unstable in the present restricted sense. If α is negative, the amplitudes of vibration continuously decrease and finally die out. These various types of motion have already been represented in fig. 136. It follows at once that an examination of the roots of

$$A\lambda^4 + B\lambda^3 + C\lambda^2 + D\lambda + E = 0$$

from this point of view will determine whether the machine is stable to exceedingly small longitudinal disturbances. Various methods have been elaborated to facilitate such examination, and these will be here shortly noted. These criteria also apply when two or more roots of (33) are equal but the form of the solution is not given by (34), as can easily be verified.

§ **13.** (*a*) An initial test may be rapidly applied by the following method. Writing

$$y = A\lambda^4 + B\lambda^3 = -C\lambda^2 - D\lambda - E,$$

the required roots if real will be given by the intersection of the two curves (I) and (II) in fig. 140, where

$$y = A\lambda^4 + B\lambda^3$$

and $$y = -C\lambda^2 - D\lambda - E$$

are plotted.

The case illustrated is that of two real roots $\lambda = \alpha$ and $\lambda = \beta$ say obtained from the diagram. It follows that $(\lambda - \alpha)$ and $(\lambda - \beta)$ must be factors to the order of approximation of the

drawing, in the biquadratic, and by dividing out the other quadratic factor is at once found and solved.

(*b*) The following approximation to the biquadratic in λ is found for most practical purposes to be sufficiently close, viz.

$$(\lambda^2 + B\lambda + C)[\lambda^2 + (D/C - BE/C^2)\lambda + E/C] = 0 \ldots (35),$$

taking $A = 1$, and the roots follow at once by the solution of each of the quadratic equations. It may be stated that this is sufficiently accurate in general if

B is not greater than C,

E is not greater than $C^2/20$,

and D is not greater than $BC/20$.

These conditions are usually satisfied by most machines.

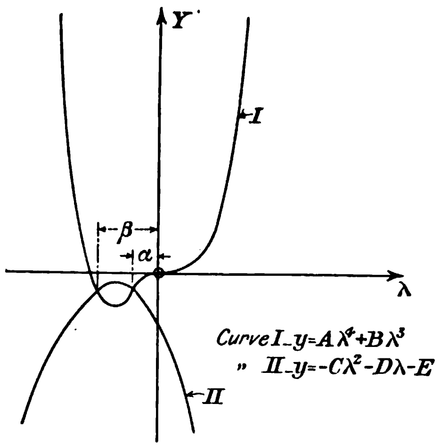


FIG. 140

(*c*) It can be shown from the theory of equations, that the biquadratic will furnish values of λ corresponding to stable motion provided A, B, C, D and E are all positive and Routh's discriminant, viz. $BCD - AD^2 - EB^2$, is positive. This is really the condition that the roots of the equation shall all have their real parts negative. For a complete discussion and proof of these conditions, and other methods of determining approximately the roots of the equation, the student is referred to a text book on the theory of equations.

The interpretation of the foregoing conclusions from the point of view of design will be reserved for the next chapter.

§ **14.** *Asymmetric Disturbances.* **The method that has been developed for the treatment of longitudinal disturbances on the equilibrium motion of the machine in symmetrical straight line flight, may be applied with scarcely any modification to the more general case where the disturbance, still small, is not restricted to lie in the plane of symmetry. The axes of reference, fixed as before in the body, will be the longitudinal axis of the machine, OX, i.e. a line through the centre of gravity originally parallel to the path of equilibrium flight, an axis OZ perpendicular to this in the plane of symmetry, and one OY through the centre**

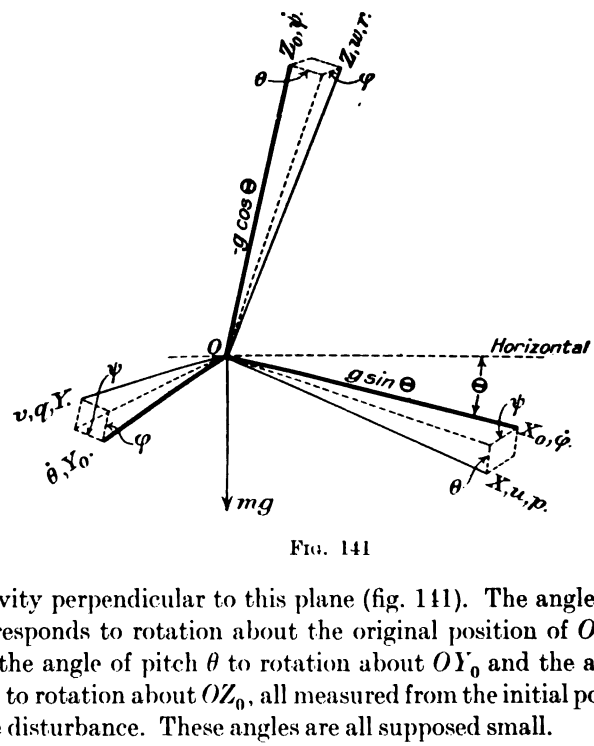


FIG. 141

of gravity perpendicular to this plane (fig. 141). The angle of roll ϕ corresponds to rotation about the original position of OX, say OX_0, the angle of pitch θ to rotation about OY_0 and the angle of yaw ψ to rotation about OZ_0, all measured from the initial positions before disturbance. These angles are all supposed small.

As a result of the disturbance, the machine ceases to occupy its previous attitude of equilibrium, and has communicated to it angular velocities, $\dot{\phi}$, $\dot{\theta}$ and $\dot{\psi}$ measured about the original axes OX_0, OY_0, OZ_0, together with small changes in the linear velocity components of the machine. These are equivalent to p, q, r, angular velocities about, and u, v, w, linear velocities along the

new system of axes OX, OY, OZ. By simple resolution of angular velocities it will be seen that the following relations exist between these quantities, neglecting terms of the second order,

$$d\phi/dt \text{ or } \dot{\phi} = p \quad \text{......................(36)},$$

$$\dot{\theta} = q \quad \text{......................(37)},$$

$$\dot{\psi} = r \quad \text{......................(38)}.$$

Proceeding on the lines outlined in the earlier discussion, the equations of motion of the centre of gravity of the machine in the general case take the form

$$m\left(\frac{dU}{dt} + WQ - VR\right) = mX \text{..............(39)},$$

$$m\left(\frac{dV}{dt} + UR - WP\right) = mY \text{..............(40)},$$

$$m\left(\frac{dW}{dt} + VP - UQ\right) = mZ \text{..............(41)},$$

where U, V, and W are the linear velocities and P, Q, and R the angular velocities about the new axes OX, OY, OZ. The equations for the rotations about the centre of gravity are

$$\frac{dH_1}{dt} - RH_2 + QH_3 = AL \text{.................(42)},$$

$$\frac{dH_2}{dt} - PH_3 + RH_1 = BM \text{..............(43)},$$

$$\frac{dH_3}{dt} - QH_1 + PH_2 = CN \text{..............(44)},$$

where

H_1 = angular momentum or moment of momentum of the machine about the new axis OX,

$$H_1 = PA - QF - RE,$$
$$H_2 = QB - RD - PF,$$
$$H_3 = RC - PE - QD.$$

A, B and C are the moments of inertia about the axes OX, OY and OZ ($= mK_a^2$, mK_b^2, mK_c^2), and D, E, F the corresponding products of inertia (mK_d^2, etc.). In the present problem D and F are zero. The foregoing six equations apply quite generally without restriction. Narrowing these down to the problem under discussion, the following set of equations is obtained as

applicable subsequent to the disturbance, neglecting second order terms. Substituting $U_0 + u$ for U, v for V, w for W, p for P, etc., since V_0, W_0, etc. are zero, then as before

$$m\frac{du}{dt} = mX \quad \text{..................(45)},$$

$$m\left(\frac{dv}{dt} + U_0 r\right) = mY \quad \text{..................(46)},$$

$$m\left(\frac{dw}{dt} - U_0 q\right) = mZ \quad \text{..................(47)},$$

$$A\frac{dp}{dt} - E\frac{dr}{dt} = AL \quad \text{..................(48)},$$

$$B\frac{dq}{dt} = BM \quad \text{..................(49)},$$

$$C\frac{dr}{dt} - E\frac{dp}{dt} = CN \quad \text{..................(50)}.$$

The forces mX, mY, mZ depend for their values on the components of gravity and like AL, BM, CN upon the resisting forces on the machine. The components of gravity along these three directions are easily seen to be (fig. 141)

$$g \sin\Theta_0 + g\theta\cos\Theta_0 \quad \text{in the } X \text{ direction ...(51)},$$

$$-g\psi\sin\Theta_0 - g\phi\cos\Theta_0 \quad \text{in the } Y \text{ direction ...(52)},$$

$$-g\cos\Theta_0 + g\theta\sin\Theta_0 \quad \text{in the direction } Z \text{ ...(53)}.$$

Expanding mX in a similar manner to that carried through in equation (23), but including the propeller thrust and the gravity term in the longitudinal force, we get

$$mX = (mg\sin\Theta_0 + mX_0 - T_0) + m(\theta g\cos\Theta_0 + uX_u + vX_v + wX_w + pX_p + qX_q + rX_r) \text{ ...(54)},$$

and similar expressions for mY and mZ. The momental terms on the right-hand side of equation (42) can likewise be thrown into the following form, L_0, etc. being zero:

$$AL = A(uL_u + vL_v + wL_w + pL_p + qL_q + rL_r) \text{...(55)},$$

$$BM = \text{etc.}$$

The equations for equilibrium before the disturbance can be obtained from (45) to (55) by making u, v, w, p, q, r, θ, ϕ, ψ, all vanish. The conditions thus obtained allow of the omission of the finite terms from the equations for the small disturbance.

A further simplification may be introduced by noting that certain of the resistance derivatives must of necessity vanish. From symmetry all derivatives of X, Z and M with respect to v, p, and r and Y, L and N with respect to u, w and q are all zero. M_u moreover as before also vanishes. The equations (45) to (50) for small oscillations under an asymmetric disturbance can now be represented in the comparatively simple form

$$\begin{aligned}
&\frac{du}{dt} - uX_u \qquad - wX_w \qquad - qX_q - \theta g \cos\Theta_0 &&= 0,\\
&\frac{dv}{dt} - vY_v \qquad - pY_p + \phi g \cos\Theta_0 \qquad + r(U_0 - Y_r) + \psi g \sin\Theta_0 &&= 0,\\
&- uZ_u \qquad + \frac{dw}{dt} - wZ_w \qquad - q(U_0 + Z_q) - \theta g \sin\Theta_0 &&= 0,\\
&- vL_v \qquad + \frac{dp}{dt} - pL_p \qquad - \frac{E}{A}\frac{dr}{dt} - rL_r &&= 0,\\
&- wM_w \qquad + \frac{dq}{dt} - qM_q &&= 0,\\
&- vN_v \qquad - \frac{E}{C}\frac{dp}{dt} - pN_p \qquad + \frac{dr}{dt} - rN_r &&= 0.
\end{aligned}$$

Remembering that $p = d\phi/dt$, $q = d\theta/dt$ and $r = d\psi/dt$ as already derived, the above six differential equations suffice to determine the six variables u, v, w, θ, ϕ, ψ as functions of t. Proceeding as in the case of longitudinal disturbances, this can most conveniently be done by supposing these quantities proportional to $e^{\lambda t}$, and eliminating t. The following determinant is obtained:

$$\begin{vmatrix}
\lambda - X_u, & 0, & -X_w, & 0, & -\lambda X_q - g\cos\Theta_0, & 0\\
0, & \lambda - Y_v, & 0, & -\lambda Y_p + g\cos\Theta_0, & 0, & \lambda(U_0 - Y_r) + g\sin\Theta_0\\
-Z_u, & 0, & \lambda - Z_w, & 0, & \begin{matrix}-\lambda(U_0 + Z_q)\\ -g\sin\Theta_0,\end{matrix} & 0\\
0, & -L_v, & 0, & \lambda^2 - \lambda L_p, & 0, & -\frac{E}{A}\lambda^2 - \lambda L_r\\
0, & 0, & -M_w, & 0, & \lambda^2 - \lambda M_q, & 0\\
0, & -N_v, & 0, & -\frac{E}{C}\lambda^2 - \lambda N_p, & 0, & \lambda^2 - \lambda N_r
\end{vmatrix} = 0.$$

This, as can easily be verified, reduces to the following two determinants:

$$\begin{vmatrix}
\lambda - X_u, & -X_w, & -\lambda X_q - g\cos\Theta_0\\
-Z_u, & \lambda - Z_w, & -\lambda(U_0 + Z_q) - g\sin\Theta_0\\
0, & -M_w, & \lambda^2 - \lambda M_q
\end{vmatrix} = 0 \qquad \text{.........(56)};$$

and

$$\begin{vmatrix} \lambda - Y_v, & -\lambda Y_p + g\cos\Theta_0, & \lambda(U_0 - Y_r) + g\sin\Theta_0 \\ -L_v, & \lambda^2 - \lambda L_p, & -\frac{E}{A}\lambda^2 - \lambda L_r \\ -N_v, & -\frac{E}{C}\lambda^2 - \lambda N_p, & \lambda^2 - \lambda N_r \end{vmatrix} = 0 \quad \text{.........(57).}$$

§ **15.** A remarkable conclusion can be immediately deduced. The determinant (56) is identical with that derived in the discussion on longitudinal disturbances. Its terms do not involve any of the quantities concerned with lateral motion. Determinant (57) on the other hand is quite independent of any quantity that arises from longitudinal disturbances. It follows at once that for symmetrical straight line flight the problem of longitudinal stability is quite unaffected by the question of whether the machine is laterally stable or not.

§ **16.** A discussion of this second determinant similar to that carried through for the first will decide the question of lateral stability. Expanding, we obtain

$$A_1\lambda^5 + B_1\lambda^4 + C_1\lambda^3 + D_1\lambda^2 + E_1\lambda + F_1 = 0 \text{......(58),}$$

where

$$A_1 = 1 - K_E^4/K_A^2K_C^2,$$

$$B_1 = Y_v \cdot \frac{K_E^2}{K_A^2} \cdot \frac{K_E^2}{K_C^2} - \begin{vmatrix} 1, & L_r \\ -K_E^2/K_C^2, & N_r \end{vmatrix} - \begin{vmatrix} L_p, & -K_C^2/K_A^2 \\ N_p, & 1 \end{vmatrix},$$

$$C_1 = \begin{vmatrix} Y_v, & Y_p, & 0 \\ L_v, & L_p, & -K_E^2/K_A^2 \\ N_v, & N_p, & 1 \end{vmatrix} - \begin{vmatrix} Y_v, & 0, & Y_r - U \\ L_v, & -1, & L_r \\ N_v, & K_E^2/K_C^2, & N_r \end{vmatrix} + \begin{vmatrix} L_p, & L_r \\ N_p, & N_r \end{vmatrix},$$

$$D_1 = -\begin{vmatrix} Y_v, & Y_p, & Y_r - U \\ L_v, & L_p, & L_r \\ N_v, & N_p, & N_r \end{vmatrix} + g\cos\Theta_0 \begin{vmatrix} L_v, & -K_E^2/K_A^2 \\ N_v, & 1 \end{vmatrix} + g\sin\Theta_0 \begin{vmatrix} L_v, & 1 \\ N_v, & K_E^2/K_C^2 \end{vmatrix},$$

$$E_1 = -g\cos\Theta_0 \begin{vmatrix} L_v, & L_r \\ N_v, & N_r \end{vmatrix} + g\sin\Theta_0 \begin{vmatrix} L_v, & L_p \\ N_v, & N_p \end{vmatrix},$$

$$F_1 = 0,$$

and where $mK_E^2 = E, \quad mK_C^2 = C, \quad mK_A^2 = A.$

One of the roots of the quintic equation in λ is evidently zero, and corresponds to the fact that the machine is directionally neutral. The resulting quartic

$$A_1\lambda^4 + B_1\lambda^3 + C_1\lambda^2 + D_1\lambda + E_1 = 0 \quad \text{.........(59)}$$

will determine the stability properly so called.

§ **17.** The various tests for the roots that have been applied in the case of longitudinal stability can similarly be used in treating this quartic, the coefficients of which as before can generally be determined by experiment. An approximate form for equation (58) extremely useful in practice is

$$(\lambda + E_1/D_1)\left(\lambda + \frac{B_1^2 - C_1}{B_1}\right)\left\{\lambda^2 + \left(\frac{C_1}{B_1} - \frac{E_1}{D_1}\right)\lambda + \frac{B_1 D_1}{B_1^2 - C_1}\right\} = 0 \quad \text{.........(60)},$$

where E_1 is not greater than $1/20\, B_1$ or D_1, and $B_1 D_1 - C_1^2$ is not greater than $1/20\, C_1^2$, conditions which are usually satisfied fairly closely. Frequently it is found that $K_E{}^2$ is negligible.

The interpretation of the foregoing results in terms of design is reserved for the next chapter.

§ **18.** *General Remarks.* Although the discussion has so far been restricted to the stability of symmetrical straight line flight the method can equally well be applied to the problem next in importance and simplicity, steady circular and helical flight. Under these circumstances the angular velocities about the axes become finite in value and consequently, in equations (39 etc.), there would now require to be substituted for P, Q and R,

$$P_0 + p, \quad Q_0 + q \quad \text{and} \quad R_0 + r.$$

At the same time the system of equations from which the determinants are derived would be modified in form. This can be easily obtained by similar resolution to that already carried through. From these θ, ϕ and ψ in terms of p, q and r are derived and substituted in the modified form of the gravity components (51, etc.). The elimination of u, v, w, p, q and r from these furnishes an octic in λ. The coefficients in this expression are, however, now extremely complicated, not merely on account of the fact that the

eighteen resistance derivatives which vanished previously for symmetrical straight line flight have finite values difficult of experimental determination, but all the derivatives now depend for their values upon the angular velocities P_0, Q_0, R_0 in addition to U_0, V_0, W_0. Even in the case of steady straight line flight if terms be introduced involving the gyroscopic effect of the rotating parts the solution is further complicated and results in an octic in λ.

CHAPTER XIII

STABILITY—EXPERIMENTAL

§ **1.** The mathematical theory of the stability of an aeroplane in steady straight line flight, outlined in the previous chapter, indicates that the problem reduces itself to a discussion of the two equations

$$A\lambda^4 + B\lambda^3 + C\lambda^2 + D\lambda + E = 0,$$
$$A_1\lambda^4 + B_1\lambda^3 + C_1\lambda^2 + D_1\lambda + E_1 = 0,$$

for longitudinal and lateral disturbances respectively, the coefficients being given by, since $K_E{}^2$ is nearly equal to zero,

$$A = 1 \qquad\qquad A_1 = 1$$

$$B = -(M_q + X_u + Z_w) \qquad\qquad B_1 = -(N_r + L_p + Y_v)$$

$$C = \begin{vmatrix} Z_w, & U_0 + Z_q \\ M_w, & M_q \end{vmatrix} + \begin{vmatrix} X_u, & X_q \\ 0, & M_q \end{vmatrix} + \begin{vmatrix} X_u, & X_w \\ Z_u, & Z_w \end{vmatrix} \qquad C_1 = \begin{vmatrix} Y_v, & Y_p \\ L_v, & L_p \end{vmatrix} + \begin{vmatrix} Y_v, & Y_r - U_0 \\ N_v, & N_r \end{vmatrix} + \begin{vmatrix} L_p, & L_r \\ N_p, & N_r \end{vmatrix}$$

$$D = -\begin{vmatrix} X_u, & X_w, & X_q \\ Z_u, & Z_w, & U_0 + Z_q \\ 0, & M_w, & M_q \end{vmatrix} - gM_u \sin\Theta_0 \qquad D_1 = -\begin{vmatrix} Y_v, & Y_p, & Y_r - U_0 \\ L_v, & L_p, & L_r \\ N_v, & N_p, & N_r \end{vmatrix} + (L_v \cos\Theta_0 + N_v \sin\Theta_0)$$

$$E = gM_w \begin{vmatrix} X_u, & \cos\Theta_0 \\ Z_u, & \sin\Theta_0 \end{vmatrix} \qquad E_1 = -g\cos\Theta_0 \begin{vmatrix} L_v, & L_r \\ N_v, & N_r \end{vmatrix} + g\sin\Theta_0 \begin{vmatrix} L_v, & L_p \\ N_v, & N_p \end{vmatrix}.$$

§ **2.** *General Discussion of Derivatives.* The coefficients are functions of the various parts and settings of the machine, the speed of flight and density of the air. For each flight path, therefore, these quantities will assume a special value, and an investigation into the stability or instability of a particular machine would require to cover variations in these for the whole flying range and height. It is conceivable at the outset that although stability might exist at a particular height and speed, it might not persist at a greater altitude and higher speed of flight. An inspection of the coefficients for dependence on density indicates that their relative magnitude will alter with variation in this quantity, since some of them are proportional to ρ, others to ρ^2 and yet others are mixed functions of this quantity as far as ρ^3. Remembering that at altitudes not infrequently attained, the

density diminishes to roughly two-thirds its ground value, it will be seen that this variation might become of importance and would therefore demand in such cases a re-estimation of the roots of the biquadratic.

In general, however, for the same attitude, it is found that even for such changes in density, the character of the stability or otherwise is unaffected. The effect of change of speed on the other hand, involving as it does a difference in the attitude of the machine, is of much deeper consequence, and an investigation of the stability for various speeds within the flying range must be carried through.

The resistance derivatives X_u, etc., and the rotary derivatives M_q, etc., are ultimately only to be determined accurately for a particular machine by a special experiment, but the actual carrying out of this experiment may be frequently avoided by a calculation of these quantities, based on the magnitudes of the parts most responsible in producing the effects associated with each derivative. The degree to which such parts of the machine as the elevators and tail plane, for example, affect the values of these quantities, and the nature of the motion resulting from the disturbance, can be clearly understood from the following consideration which, at the same time, illustrates certain limitations of the investigations of the previous chapter.

§ **3.** *Significance of Tail Plane and Elevators in Stability.* For a given machine it has already been shown that the elevator setting determines the angle of attack in equilibrium and consequently also the speed of straight line flight (cf. § 3, Chapter XI). If, however, the restriction regarding equilibrium be removed it is evident that to each elevator setting and angle of attack there will correspond a pitching moment. A series of curves of the type shown in fig. 142 can accordingly be constructed showing the variation in pitching moment coefficient with angle of attack for a series of settings β_1, β_2, etc. of the elevator, where the pitching moment coefficient is defined as the pitching moment divided by ρV^2. The points a_1, a_2, a_3, etc. corresponding to no pitching moment are positions of equilibrium flight for the angles of attack oa_1, oa_2, etc., with elevator settings β_1, β_2, etc. Strictly speaking, it is at these points that the stability is to be investigated. In the first place, a real distinction exists between the nature of the

equilibrium flight at the point a_1 and that at a_6, where the slope of the curve on crossing the horizontal axis is of opposite sign to that of the former. At a_1 a slight increase in angle of attack brings into being a negative pitching moment producing a tendency for the angle of attack to revert back to that at a_1. A slight decrease in angle of attack produces a positive pitching moment, tending again to throw the machine back to its original position. As far as the present quèstion is concerned, therefore, flight in the position a_1 is stable. At a_6, on the other hand, a similar consideration indicates that an increase or decrease in the angle of attack gives rise to a pitching moment in both cases throwing the machine further from its position of equilibrium. a_6 is therefore a position

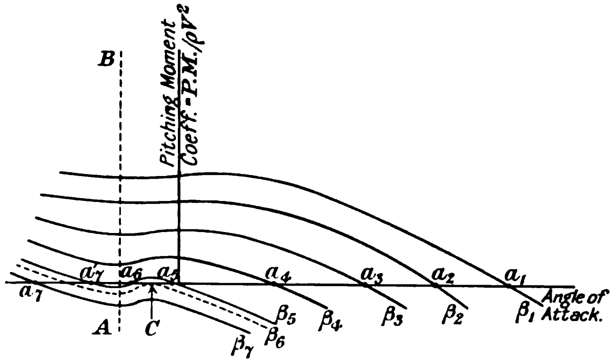


FIG. 142

of instability. It would appear that for any of the elevator settings β_1, β_2, etc., for any position in which the machine was placed, the angle of attack would change continuously until it occupied the positions a_1, a_2, etc., about which it would oscillate until it finally settled. For an elevator setting of the type β_7, whose pitching moment curve does not cross the horizontal axis until a_7 is reached corresponding to an angle of attack below the "no lift position" AB, the machine would continuously change its attitude under the effect of this negative pitching moment, and passing the no lift position would finally adopt the attitude determined by the angle of attack at a_7 corresponding to upside-down flight. The transition to this type of possible flight will be affected at the elevator setting β_6 which gives a curve just touching the horizontal axis at C.

Suppose the pilot is in equilibrium flight at high speed with an elevator setting β_5 and angle of attack of course that at a_5. For disturbances less than a certain magnitude the flight is stable. A larger disturbance, however, great enough to throw the machine up to, or beyond, the position a_6 would drive it finally to occupy the attitude of equilibrium a_7'. In practice this disturbance might be a gust of wind which thus suddenly pitches the machine over to a large negative angle of attack, and before there is any possibility of recovery on the part of the pilot, there comes into play a downward wind force on the aeroplane which together with the weight imparts an immediate downward acceleration approximately equal to $2g$. Unless there be sufficient height available to afford opportunity for a rapid alteration of the elevator setting, etc., disastrous consequences would ensue.

The weakness of the mathematical analysis of stability is in this instance well brought out, for it appears from the discussion of the preceding chapter that under these circumstances the machine occupies a position of stability for small disturbances whereas in actual fact for larger disturbances it is unstable.

§ 4. *Estimation of Dimensions of Tail.* For a clear understanding as to how such instability can be avoided by accurate design of the parts responsible, it is of importance to analyse exactly the manner in which the pitching moment curves of fig. 142 arise. It is evident that two factors are of primary importance in contributing towards the effect under discussion. A reference to fig. 66, Chapter VI, indicates that, as the angle of attack of the aerofoil increases, the centre of pressure moves forward, and consequently a pitching moment about the centre of gravity arises, the variation of whose coefficient with angle of attack is given generally by fig. 143, curve A.

The only other part of the machine that produces a pitching moment of any consequence is the tail. In value this moment is approximately the product of the force upon the tail, and the distance of the centre of the tail from the centre of gravity of the machine. Or,

$$\text{Pitching moment coefficient due to tail} = Al.T_t,$$

where A is the area of the tail, l its distance from the centre of

gravity and T_l the lift coefficient of the tail, or lift on tail $\div \rho V^2$. Since T_l is found to be practically proportional to the angle of attack and is in fact usually $- 0{\cdot}025 (\alpha - \beta)$ where β is the angle of no lift and α the angle of attack of the tail, in degrees, this may be written $- 0{\cdot}025Al (\alpha - \beta)$. A change in elevator setting will correspond to a change in β only, while α can only vary with the angle of attack of the machine. The discussion on downwash, in fact, in Chapter VII indicates that α varies practically at a rate equal to one-half that of the angle of attack of the main planes. It is thus a simple matter to plot the pitching moment coefficient due to the tail on the same base as that due to the main planes, fig. 143. These give a series of parallel lines for various elevator settings, the slope of which will be $- 0{\cdot}025Al/2$ (the straight line).

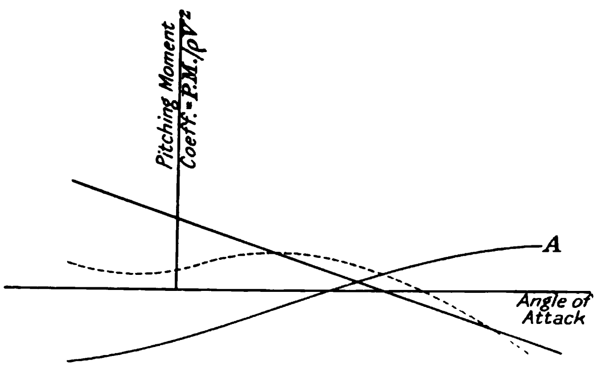


FIG. 143

By adding the ordinates of curve A and the latter algebraically, at the same angle of attack the final pitching moment coefficient curve, shown dotted, for the whole machine is obtained very approximately.

The previous discussion and interpretation of this curve indicated that in order to avoid instability no crossings of the horizontal axis must occur at a positive slope, as for example those at a_6, etc., fig. 142, for some distance on each side of the flying range of angle of attack. This will evidently be achieved, if the slope of the pitching moment curve due to the tail is everywhere negative, and greater than the maximum positive slope of the pitching moment curve due to the main planes. For in such an event, the slope of the final curve obtained by adding ordinates will be everywhere negative, and therefore can cross the zero axis

only once. An approximate rule for setting a lower limit to the leverage of the tail area follows at once. The main planes having been selected according to the principles formulated in Chapters V and VI, and their dimensions and the position of the centre of gravity of the machine determined, a knowledge of the variation of centre of pressure and lift and drag coefficients with angle of attack will suffice for an estimation of the pitching moment coefficient. If σ be the maximum slope of this curve, then $0{\cdot}0125Al$ must be greater than σ in order to avoid this type of instability.

§ 5. This illustration demonstrates the manner in which the component parts of the machine operate together to produce or avoid instability, and provides a moderately exact method for determining the leverage of the tail required under certain circumstances. For lateral effects a similar process might be adopted, but in this case even when the calculations thus indicated have been carried through, it does not follow that if the aeroplane be constructed to satisfy these demands it will be stable. A certain amount of interference between the various parts might and does take place, sufficient in certain cases to convert an otherwise apparently stable machine into an unstable one. Ultimately recourse must in all cases be had to an examination of the two biquadratics for a final test, but it is important that the region of investigation should be extended for some distance beyond the normal flying range to allow in some measure for the effect of finite disturbances.

§ 6. *Experimental Determination of Force and Rotary Derivatives.* Before the biquadratic can be properly built up, the values of the coefficients must be derived from a knowledge of the force derivatives and the rotaries. Generally it is possible to estimate these roughly from the dimensions of the parts of the machine in question, but more accurate and reliable information can only be obtained by wind tunnel experiments on the complete model. This would bring to bear on the flow past the various parts that determine the values of these derivatives the interference due to the existence of the remainder of the machine. Any method of calculation would of necessity involve only a very inaccurate correction for this effect, and to that extent at least would be unreliable.

As has already been explained the various force derivatives can be expressed in terms of the slopes of the lift, the drag, the cross wind force, the pitching moment, the yawing moment and the rolling moment curves, plotted against velocity, angle of attack, and of yaw. The calculations must then be carried through for stability at a particular angle of attack within the range of these experimental determinations.

The longitudinal derivatives have already been treated in some detail, and for convenience of estimation simple expressions for these and the corresponding laterals are tabulated below.

Longitudinals	*Laterals*
$X_u = -\frac{gU_0'}{W} \cdot 2d_0$	$Y_u = 0$
$X_w = \frac{gU_0'}{W}\left(l_0 - \frac{\partial d_0}{\partial \alpha}\right)$	$Y_v = \frac{gU_0'}{W}\left(\frac{\partial c_0}{\partial \psi} - d_0\right)$
$Z_u = -\frac{2gl_0U_0'}{W}$	$L_u = 0$
$Z_w = -\frac{gU_0'}{W}\left(\frac{\partial l_0}{\partial \alpha} + d_0\right)$	$L_v = \frac{gU_0'}{W} \cdot \frac{\partial l_0'}{\partial \psi} \cdot \frac{1}{K_A^2}$
$M_u = 0$*	$N_u = 0$
$M_w = -\frac{gU_0'}{W} \cdot \frac{\partial m_0}{\partial \alpha} \cdot \frac{1}{K_B^2}$	$N_v = \frac{gU_0'}{W} \cdot \frac{\partial n_0'}{\partial \psi} \cdot \frac{1}{K_C^2}$

where

$L_0 = l_0U_0^2$ = total lift on complete machine,

$D_0 = d_0U_0^2$ = total drag on complete machine,

$M_0 = m_0U_0^2$ = pitching moment on complete machine,

$C_0 = c_0U_0^2$ = total cross-wind force on complete machine,

$L_0' = l_0'U_0^2$ = rolling moment on complete machine,

$N_0' = n_0'U_0^2$ = yawing moment on complete machine,

K_A = radius of gyration about OX,

K_B = ,, ,, ,, ,, OY,

K_C = ,, ,, ,, ,, OZ,

the suffixes referring to the values of the quantities considered

* Only true if the effect of the propeller be neglected.

when the controls are fixed for normal flying position at the angle of attack under investigation.

Of the nine rotary derivatives M_q, N_r, N_p, L_r, X_q, L_p, Z_q, Y_p, Y_r, the latter seven are exceptionally difficult of experimental determination, and any such method that could be devised would probably involve errors sufficiently great to vitiate the results. Fortunately, however, these quantities are not of sufficient importance, where they enter into the coefficients, to demand more than a very approximate idea of their magnitude, and it suffices to estimate this in general by calculation. The method to be followed in these cases will be indicated shortly. Both M_q and N_r on the other hand are very easily derived by a simple experiment in the wind channel. In the case of M_q the model is supported on its side by a vertical spindle, through the centre of gravity of the machine, and attached to the channel balance constrained to rotate about the axis of the spindle. By means of a spring a slight angular displacement communicated to the machine causes it to oscillate about the position of equilibrium corresponding to the angle of attack. From a comparison of the rate of damping with and without the wind, M_q is easily calculated using the constants of the balance.

In the same way by supporting the model in the horizontal position in the proper attitude, N_r is determined.

To complete the discussion of the derivatives there remains to indicate how those quantities, not conveniently obtained by experiment, can be estimated from a knowledge of the dimensions of the various parts of the machine. Incidentally there will be shown the effect of variation in these dimensions upon the more important derivatives.

§ 7. *Effect of Various Parts. Illustration of approximate calculation.* Sufficient has already been said to make clear the stabilising function of the tail. Its importance lies in the fact that it tends to increase M_w which is proportional to the rate of increase of the pitching moment with an increase in normal velocity. Other quantities such as M_q are likewise affected by this part, but M_w overshadows in importance all other factors as can easily be verified by an examination of the relative magnitudes of the terms entering into the biquadratic for the longitudinals.

There are two principal components of the machine that govern and practically determine the lateral stability. These are (*a*) the fin, and (*b*) the dihedral angle of the wings.

Fin. An elementary consideration will quickly separate out the terms that are affected by the presence of the fin, and even generally to what extent they are affected. A force upon the fin, neglecting the drag, will modify the values of the force Y and of the moments L and N. The change in Y being roughly proportional to the area of the fin relative to the longitudinal cross-section of the machine will be comparatively small, and is in fact negligible. The variation in L, the moment about the longitudinal axis due to the fin, has as leverage a comparatively small distance and therefore may likewise be discarded in the face of more important variations in L, such as that due to the wings themselves. There remains only to be considered the changes in the moments about the normal axis. Of these N_v is of outstanding importance.

N_v, which comes into existence when the wind yaws relative to the axis of the machine, is not easy to estimate. It is affected by the body as well as the fin and rudder, the actual amount of the body interference being determined by the proportions behind and before the centre of gravity. It will be increased by an increase in fin area, or by placing the fin further back from the centre of gravity. The importance of this term will shortly be seen when it will be shown that N_v tends to predominate in certain of the coefficients of the biquadratic, determining their sign, and thus indicating the effect on stability. The chief terms affected, it will be seen, are E_1 and D_1, the latter written very approximately as

$$D_1 = -Y_v \begin{vmatrix} L_p, & L_r \\ N_p, & N_r \end{vmatrix} + U_0 \begin{vmatrix} L_v, & L_p \\ N_v, & N_p \end{vmatrix} + g\cos\Theta . L_v.$$

Dihedral. In many respects the dihedral may be more easily considered by supposing substituted an equivalent fin area situated above the centre of gravity. The only term which is to any extent affected is L_v. As already indicated this term is scarcely modified at all by the body and fin proper, and its value is almost entirely dependent on the dihedral angle. An increment v alters the angle of attack on one wing from a to $a + \frac{v}{U}\beta$ and on the other to $a - \frac{v}{U}\beta$, where β is half the

dihedral angle. Approximately, therefore, the moment is increased by

$$mg \cdot \frac{\beta}{a} \cdot \frac{v}{U'} \cdot \frac{l}{2},$$

l being the length of each wing, and this, for all practical purposes, may be used for vL_v.

Summing up it appears that the effect of the fin will be most apparent in the variation it causes in N_v while the dihedral on the other hand makes itself evident by the changes it involves in L_v. It will be most convenient to trace the effects of these two elements through their representative coefficients.

The effect of these derivatives can be seen immediately from the fact that N_v, if positive, tends to make D_1 negative, and therefore acts towards instability. If on the other hand this derivative is large and negative, E_1 tends to become negative and again acts towards instability. There exists between these two extremes a range of values of N_v satisfying the conditions for stability as far as both D_1 and E_1 are concerned.

From an examination of E_1, it can be seen that N_v becomes a factor when L_v is zero. It follows that, other things remaining practically unchanged, the sign of E_1 in this case would be determined by N_v. If these quantities be positive, D_1, as already seen, becomes negative and consequently indicates instability. To obviate this difficulty, L_v must attain a positive value. Interpreted physically this demands the existence of a dihedral angle on the wings.

The terms X_q, Z_q, Y_p, Y_r are for most machines small in magnitude and, as long as they remain so, scarcely affect the stability conditions. They will accordingly not be treated. L_p, N_p, L_r on the other hand do attain some significance in the equations, and their method of estimation can be briefly indicated. The angular rotation p is equivalent as far as L is concerned to a change in the angle of attack for each element of the wing. The new normal force, or approximately lift, at each position on the wing can thus be obtained and the moment mpL_p derived by summation. In the same way the new longitudinal force will give moments about the Z-axis, their sum constituting mpN_p. For the estimation of L_r it must be remarked that the increase in velocity of each element of the wings, due to the increase in angular velocity r, causes a change in the lifting force at each point. Each

element thus contributes towards the moment about the longitudinal axis, and is estimated by summation.

The following typical values of the derivatives are given for reference.

Ft.-sec. units.

$$m = \text{wt. in lbs.}/g = 40, \quad K_A{}^2 = K_B{}^2 = 25 \quad \text{and} \quad K_C{}^2 = 35.$$

Longitudinal derivatives, etc.

$$X_u = -0{\cdot}14, \quad Z_u = -0{\cdot}80, \quad M_u = 0,$$
$$X_w = 0{\cdot}19, \quad Z_w = -2{\cdot}89, \quad M_w = 0{\cdot}106,$$
$$X_q = \pm 0{\cdot}5, \quad Z_q = 9{\cdot}0, \quad M_q = -8{\cdot}4.$$

Lateral derivatives.

$$Y_v = -0{\cdot}25, \quad L_v = 0{\cdot}0332, \quad N_v = -{\cdot}015,$$
$$Y_p = 1, \quad L_p = -8, \quad N_p = -{\cdot}57,$$
$$Y_r = -3, \quad L_r = 2{\cdot}6, \quad N_r = -1{\cdot}05^*.$$

Physical Interpretation. Analysis of Types of Oscillations Originated during Flight

§ 8. (i) *Longitudinals.* The biquadratic may be written in the approximate form

$$[\lambda^2 + B\lambda + C]\,[\lambda^2 + (D/C - BE/C^2)\,\lambda + E/C] = 0,$$

and by considering each of these factors in turn it is possible to correlate each part of the motion with the portion of the machine concerned in producing it. Some of the terms in the coefficients in this equation are found in practice to be so small that to a sufficient degree of accuracy it is possible to write

$$A = 1,$$
$$B = -(M_q + Z_w),$$
$$C = \begin{vmatrix} Z_w, & U_0 \\ M_w, & M_q \end{vmatrix},$$
$$D = -M_q \begin{vmatrix} X_u, & X_w \\ Z_u, & Z_w \end{vmatrix} + M_w (U_0 X_u - g \sin \Theta_0),$$
$$E = -gM_w (Z_u \cos \Theta_0 - X_u \sin \Theta_0).$$

* See Report 77, Adv. Comm. for Aeronautics, 1912–13. The moments coefficients in that report are equal to the above multiplied by the square of the radius of gyration about their respective axes.

The solutions of the first factor equated to zero are

$$\lambda = \tfrac{1}{2}\{M_q + Z_w \pm \sqrt{[(M_q - Z_w)^2 + 4UM_w]}\}.$$

Both M_q and Z_w are negative and therefore instability could only arise provided the term under the radical sign is numerically greater than $M_q + Z_w$. This would involve M_w negative and in absolute value $M_q Z_w / U$. Such a state of affairs can only arise at such a point as a_6, fig. 142, where M_w is negative. Assuming this condition not satisfied, and it may be obviated by the method described in § 4, it follows that the motion may consist of two heavily damped subsidences. If the term under the radical sign is negative, which is normally the case, the motion consists of a damped oscillation of period

$$4\pi/\sqrt{[4UM_w - (M_q - Z_w)^2]} = 2 \text{ to } 3 \text{ seconds usually.}$$

The solutions of the second factor equated to zero are

$$2\lambda = -(D/C - BE/C^2) \pm \sqrt{[(D/C - BE/C^2)^2 - 4E/C]}.$$

Generally the term $D/C - BE/C^2$ is small compared with E/C, and the motion is therefore a slightly damped oscillation of approximate period $2\pi/\sqrt{(E/C)} = 15$ to 25 seconds usually.

Since the quantity under the square root is practically always negative the motion is oscillatory and instability could therefore only arise by the failure of the amplitude to decrease with time. Stability therefore demands that $(DC - BE)$ must always be positive. The type of oscillation here discussed is known as "Phugoid," and was first pointed out and explained by Lanchester.

§ **9.** (ii) *Laterals.* The biquadratic equation in λ corresponding to lateral disturbances can be written in the approximate form

$$(\lambda + E_1/D_1)[\lambda + (B_1^2 - C_1)/B_1][\lambda^2 + (C_1/B_1 - E_1/D_1)\lambda + B_1 D_1/(B_1^2 - C_1)] = 0.$$

Consider the factor

$$\lambda + (B_1^2 - C_1)/B_1 = 0.$$

In general C_1 is small compared with B_1^2, B_1 being positive, and consequently the motion represented is of the nature of a subsidence undergoing comparatively heavy damping. This latter effect is due in the main to the rolling moment originated by the rolling itself, the derivative being L_p.

The quadratic factor

$$\lambda^2 + (C_1/B_1 - E_1/D_1)\,\lambda + B_1D_1/(B_1^2 - C_1) = 0$$

can with sufficient accuracy be re-written

$$\lambda^2 + C_1\lambda/B_1 + C_1^2/B_1^2 = 0,$$

remembering that C_1/B_1^2 and E_1/D_1 have been supposed small, and provided B_1D_1 is approximately equal to C_1^2. It follows that

$$\lambda = \tfrac{1}{2}\,[-\,C_1/B_1 \pm i\sqrt{(3C_1^2/B_1^2)}].$$

This corresponds to a motion represented by

$$e^{-C_1t/2B_1}\sin\,(C_1t\,\sqrt{3}/2B_1),$$

so that the oscillation is damped, C_1 and B_1 both normally being positive, and the periodic time is $4\pi\sqrt{B_1/3C_1}$, approximately 6 seconds.

Spin and Spiral Instability. By far the most important term in the biquadratic is $\lambda + E_1/D_1 = 0$. If E_1 and D_1 be of opposite sign, instability will arise. D_1 can only become negative by N_v assuming a large positive value, as can be seen by consideration of the value of that coefficient. This would mean that the yawing moment due to side slip must be increased in the positive direction. This is equivalent to the existence of a small fin area. Such a form of instability, tending to produce rotation of the machine about a vertical axis through the centre of gravity, is usually known as "spin." Increase of the fin area appears to be the obvious remedy. Such a course however would naturally tend to diminish the rotation about a vertical axis through the centre of gravity due to side slip, and would therefore cause N_v to assume a comparatively large negative value. Reference to the expression for E_1 however indicates that this has the effect of making that quantity negative, and therefore must contribute towards instability of a different type. How this arises physically can be seen from a simple consideration. Suppose the machine is accidentally banked. Initially the centre of gravity tends to move off along a circular path, but the large fin area coming into operation swings the axis round into the new flight path. As this takes place, the outer wing, moving rapidly more than the inner, is under the action of a greater lifting force, and therefore increases the banking, tending to turn the machine still further. As before, the large fin surface swings the machine once more into the new

flight path and so on, the radius of turn diminishing continuously. During the above process the total lifting force has been constantly decreasing, involving a continual drop in altitude, and the motion ultimately developes into a rapid spiral nose dive; hence the term usually applied to this type of instability, viz. "Spiral Instability."

§ **10.** Following broadly the lines pursued in this and the previous chapter, aeronautical investigators and constructors have attained considerable success in the development of an aeroplane possessing inherent stability. It must nevertheless not be supposed that the best machine from this point of view is one possessing the maximum degree of stability, but rather one possessing the requisite measure to suit the convenience of the pilot. Too great stability results in the aeroplane turning too readily into the relative wind, and answering to every gust however slight. Such machines moreover frequently do not respond easily to their own controls.

Experience of machines in flight and an accurate study of their performance have substantiated firmly the claim that the modifications in design deduced from mathematical analysis of stability to small disturbances are practically sufficient to ensure the production of an inherently stable machine. This cannot be regarded otherwise than in the light of a signal triumph for mathematical science.

CHAPTER XIV

THE COMPLETE MACHINE

CONCLUSION

§ **1.** In the preceding chapters an attempt has been made to explain in some detail the methods and principles that must be taken as guides in the selection of the various parts of a machine. Generally an aeroplane is built to fulfil more particularly one function above others, but it does not necessarily follow that such a machine will not at the same time be comparatively efficient for other purposes. An aeroplane, for example, which has been designed for high speed above other things may at the same time rank as a fairly good climbing machine. It is outside the scope of the present book to enter into details of design, but it is proposed simply to indicate very generally where and how the principles elaborated here are to be applied. The question centres mainly around a selection of the aerofoil and the engine. In Chapter V it has been shown how the choice of the aerofoil is to be effected. A knowledge of the proposed landing speed and the probable weight of the machine then determines the wing loading and area. The body can be selected in accordance with the principles explained in Chapter VII and the setting of the wings with respect to it from the fact that, in the first place the drag of the body should be a minimum in the required flying attitude, and in the second that on landing as large an angle of incidence of the wings to the relative wind should be obtained, the skid touching the ground. For a high speed machine reduction in resistance by diminution in the number of struts is of course a desirable feature. Experience has shown that it is possible to construct a machine of approximately 2000 lbs. weight with four gap-struts only and yet be of sufficient strength. In the case of a machine devised either for weight carrying or climbing it may be advisable even to increase the number of struts in order to produce ultimately lightness in construction at the expense of a decrease

in head resistance. It may be recalled in passing that the calculations explained in Chapter VIII must be carried through for all possible types of loading that the wings, in their manœuvres, may be called upon to withstand. The same estimation must be made for the members of the body, tail, undercarriage, etc. The dimensions of the tail and the remaining controls can be determined by the methods explained in Chapter XIII, according to the degree of stability required. A final detailed discussion of stability ought to be carried through after the disposition of the various weighty portions (engine, etc.) has been decided. The propeller, whose diameter is fixed by the dimensions of the machine, should be chosen in conformity with the explanations in Chapter X, but it must be remembered that after all every propeller is practically designed by intelligent modification of a previously existing one of known performance.

§ 2. Until comparatively recent times the engine was by far the most important element in the machine and the whole construction centred round it. Now-a-days the internal combustion engine has been so far developed that its importance as the determining factor in the design of the aeroplane has diminished. The designer now has at his disposal a comparatively large number of engines of sufficiently low weight and low fuel and oil consumption per horse-power. It is clear that the exact value of these quantities will serve to limit considerably the freedom of selection of the remaining portions of the machine. The problem of weight in fact imposes very serious restrictions, since hand in hand with the question of aerodynamic suitability of parts goes the problem of the corresponding weights and their maintenance below certain limits.

§ 3. There is one further point in this connection that must not be passed over. It is tacitly assumed in general that the total resistance of the machine can be obtained by summing up the resistances of the various parts tested separately, but it is clear that mutual reference of the flow and the screening of one part by another may serve considerably to modify the accuracy of this assumption. In two particular cases where the interference effect is clear, a correction can be and usually is introduced. These cases are that of the slip stream of the propeller as it affects the

body and the planes, and the downwash from the wings upon the tail plane. In the case of models where correction is made for interference, experiment has shown that the resistance of the complete model does not differ appreciably from the sum of the resistances of its component parts. It does not follow of course that this will apply with equal truth for a complete machine, principally on account of the different degrees to which scale effect modifies the resistance of each part. Moreover there are so many small projections, not usually finding their place in the model, each one of which adds its quota to the complete resistance, that the result cannot be estimated to any considerable degree of precision. If the resistance of the complete machine under flying conditions could be accurately measured, an empiric transformation factor might be determined to assist in the passage from model to full scale. Such full scale tests are, however, extremely difficult to carry through reliably.

§ **4.** *Aeroplane testing.* To test an aeroplane scientifically, and hence to determine its performance curve for the purpose of effecting a comparison with other machines, it is necessary that the test be conducted under standard conditions or that the results be reduced to such. The forces on the wing, the resistances of the parts of the machine and the performance of the engine all depend directly on the density of the atmosphere and therefore incidentally on the temperature and pressure. It is clear that these vary continuously from time to time, and that even two consecutive tests cannot be assumed conducted under the same conditions. For purposes of standardisation, consequently, the performance—rate of climb, speed, etc.—is usually expressed as a value corrected to the standard density at a stated height. On account of the variability of atmospheric conditions as many tests as possible must be performed upon the aeroplane in order to assure sufficient accuracy.

§ **5.** There are three main series of measurements that must be determined from tests on the aeroplane in order to derive the performance curve:

1. A determination of the speed of flight,
2. A determination of the rate of climb,
3. A determination of the thrust horse-power,

and instruments have been devised for obtaining these with greater or less accuracy.

With regard to the first two it will be possible to carry these out provided efficient means are to hand to measure air speed, height, and for purposes of standardisation, density, pressure and temperature. It has already been explained in Chapter II that the pitot tube is used in ordinary wind channel work for the determination of wind speed. A similar device is normally employed here. The instrument, the pressure at the nose of which is $\frac{1}{2}\rho v^2$, is usually fitted to one of the struts and connected by a tube to a drum, and by a measurement of the deflection of the membrane the value of $\frac{1}{2}\rho v^2$ is obtained. Owing to the effect of interference and disturbance in the atmosphere by the aeroplane itself, v is not exactly the general speed of the machine, but this can be allowed for by a previous test over a measured course at ground level. It is clear that before v can be accurately calculated from this pressure measurement, the value of ρ at the height at which the experiment was conducted must be known, and this again involves a knowledge of the pressure and temperature in that position. The latter is of course determined by means of a thermometer placed on the aeroplane well away from any heating effect due to the engine. An aneroid barometer, whose function is essentially to measure height differences by the variations in pressure which are caused by such changes, unfortunately cannot be relied upon to do this accurately on account of the variations in temperature, etc. It is usually found more convenient to regard it purely as a pressure instrument, so that each height indicated on the scale corresponds to a definite pressure. From these two measurements of temperature and pressure, the density follows at once from $\rho/\rho_0 = pT_0/p_0T$, T being the absolute temperature and ρ_0, p_0 and T_0 the values of these quantities at a fixed level under standard conditions.

§ **6.** When the aneroid is used for a measurement of height, as already explained, it is subject to serious errors. In winter during cold weather it may be found to register 2000 feet too much at a height of about 15,000 feet. Such errors, which are clearly too large to be glossed over, are due mainly to the fact that the aneroid is graduated for a uniform temperature of 10° C., and for every other temperature a special scale would

require to be substituted. This depends initially upon the fact that if p_1 and p_2 be the pressures at any two points, and h_1 and h_2 corresponding heights above the zero position, then

$$h_1 - h_2 \propto (T_1 + T_2) \log p_1/p_2,$$

where T_1 and T_2 are the absolute temperatures at the points. It is from this equation that the scale in heights is obtained from the pressures for a given average temperature assumed by manufacturers to be 10° C. It is unfortunately only in exceptionally rare circumstances that the average temperature between the ground and say 20,000 feet will reach as high as this, and consequently to use the instrument as an accurate means of determining height, it is essential to correct by multiplying the height indicated by $T/283$, where T is the average temperature absolute over the range.

There is an alternative method of height and speed measurement in common use frequently adopted as a check. Two camerae obscurae are used, one pointing vertically upwards and the other sloping at some particular angle. If they be situated at some distance apart a considerable section of the sky can be projected in both cases on a horizontal plane, and an aeroplane, operating in this region, can be viewed from both points. A pilot in flight can be followed by the two observers from point to point, and by dotting his position on the charts every second, his ground speed is determined. From his position on both charts, moreover, his height is determined. This can be carried through to an accuracy of 1 in 1000. To derive the air speed accurately it is necessary to know the direction and speed of the wind at the height of the test. This is achieved by arranging for the pilot to fire off a puff of smoke and from a series of observations of its motion the ground speed of the air is likewise obtained.

§ 7. There is one further determination necessary before sufficient information is at hand to discuss the performance; that is, the thrust and thrust horse-power. If the propeller has been previously tested and if it can be assumed that during flight no warping of the blades takes place, the thrust is immediately obtained from the thrust-slip curve from the values of V, n and ρ and, of course, the pitch of the propeller. Warping, in a sense equivalent to an alteration in pitch and therefore in slip, may

however alter the thrust to a considerable extent in the working region, and it would accordingly be more convenient and more accurate, if possible, to obtain the value of this quantity, in such an event, from the efficiency-slip curve. For this purpose the engine is initially tested for horse-power output and revolutions under conditions approximating as far as possible to those of actual flight. Knowing the efficiency of the propeller from the curve the thrust horse-power can easily be derived.

§ 8. The performance curves proper may be constructed from these data and analysed by the methods adopted in preceding Chapters (V, X and XI), with special reference to such critical points as maximum rate of climb, maximum speed of horizontal flight and maximum height possible or "ceiling," but for these, special tests are usually conducted. Associated with each of these, instruments have been devised to facilitate the measurements. In particular a simple and effective apparatus is used for determining the best climbing speed of a new machine or the maximum rate of climb. It consists of a thermos flask connected to a liquid pressure gauge of small bore measuring the difference of pressure between the inside and the outside of the flask, and is at the same time open to the air by a capillary tube. During climb, since the atmospheric pressure in general is continually diminishing, air will commence to leak from the flask outwards since the pressure in the latter is greater. This will continue until the leaking takes place at such a speed that the rate of change of pressure inside the flask is equal to the rate of change of atmospheric pressure. When the maximum rate of climb is attained, therefore, the pressure difference between the inside of the flask and the external atmosphere will be a maximum, and this will be indicated by a maximum height of liquid in the pressure gauge.

It is clear that this instrument in reality measures the rate of change of pressure as the machine climbs and consequently, if the exact relation on the particular day when the instrument is used be known, between height and pressure in the region of climb, the rate of climb is likewise measured by the height of the column of liquid. It may be mentioned in passing that this same device may similarly be employed in the determination of the maximum speed of horizontal flight by using it in place of

Aeroplane Data.

Consumption trials	At R.P.M. normal	At R.P.M. economical
Total weight of aeroplane at start	2295 (1)	2376 (2)
Altitude during trial	8000 ft.	8000 ft.
Duration of trial excluding descent ...	1 hr. 13 min.	1 hr. 18 min.
Time occupied in reaching alt. of trial ...	—	17 min.
Petrol at beginning of trial	57 galls.	57 galls.
Oil " " "	7 "	7 "
Consumption of petrol per hour	11·5 "	10 "
" oil	2 "	1⅜ "
Loss of cooling water	—	—
Engine R.P.M. during trial	1350 on ground 1770 at 8000′	1440 on ground 1540 at 8000′
True air speed	79	64
Air speed indicator used	Clift	Clift
Ground speed "	—	—
" " method of measurement ...	—	—
Height of barometer	29·35″	29·4″
Temperature on ground	62° F.	57° F.

Speed trials		
Total weight of aeroplane at start	2576 (3)	2576 (3)
Altitude during trial	10 ft.	8000 ft.
Max. speed over ground, corrected for still air	101·5 M.P.H.	Mean speed = 94 M.P.H. approx. at 1830 R.P.M.
Min. " " " "	60 approx.	
Engine R.P.M. at max. speed	1975	
" " min. "	1200–1275	—
Height of barometer	—	—
Temperature	55° F.	—

Climbing trials		Height over ground ft.	Time	Rate of climb (ft. per min.)	Engine R.P.M.	Air speed by indicator
Total weight of aeroplane at start	2319 (1)	1000	1′ 30″		—	—
Consumption, petrol, galls.	12½	2000	3′ 30″		1500	60
" oil "	2	3000	5′ 50″		—	—
Duration of climb ...	60 min.	4000	8′ 30″		1500	59
Barometric height ...	29·3″	5000	11′ 0″		—	—
Temperature on ground ...	60° F	6000	13′ 30″		1500	56
Height of ground above sea level		7000	17′ 0″		—	—
Pilot lbs. / Dead wt. lbs.		8000	21′ 30″		1480	53
		9000	26′ 0″		—	—
(1) Wt. includes 173 114		10000	30′ 0″		1480	51
(2) " " 174 194		11000	36′ 30″		—	—
(3) " " 177 391		12000	46′ 0″		1460	49
(4) " " 177 134		13000	60′ 0″		1450	44·5

a spirit level. In order to attain maximum speed in horizontal flight the pilot must fly with his engine "all out," and the elevators adjusted so as to give no reading on the manometer.

The foregoing table of data obtained from an actual test may be taken as a typical specimen.

§ 9. On account of the extreme accuracy with which model experimental tests may be conducted it is clear that practically the only error that may arise is that inherent in the fact that the research is conducted upon models, viz. scale effect. To remedy this deficiency the tests conducted on full scale machines have been extended with a view to supplying information on this point. Although, as already indicated in the preceding pages, tests on the complete aeroplane supply extremely valuable information, inaccuracies non-existent in wind channel tests enter and unless adequate precautions are adopted may tend to vitiate the results. This will become clear when it is remembered that if an upward current of as little as one foot per second be unwittingly encountered the error involved in the estimation of wind drag may amount to as much as 15 per cent. Such an experimental error can clearly only be eliminated by an extremely large series of tests. Apart from the possible deficiency in respect of scale effect, model experimental work occupies a unique place on account of the control that may be obtained over the experimental conditions and the invaluable information it provides in comparative design.

GLOSSARY OF AERONAUTICAL TERMS

BASED ON THE LIST PRODUCED BY THE TECHNICAL TERMS COMMITTEE OF THE AERONAUTICAL SOCIETY OF GREAT BRITAIN

Aerofoil. A structure, analogous to the wing or tail of a bird, designed to obtain a reaction from the air approximately at right angles to the direction of its motion.

Aileron. See "Balancing Flaps."

Airscrew. Used as a generic term to include both a propeller and a tractor screw. See "Screw."

Alighting Carriage. See "Carriage."

Aneroid. Instrument used for estimating height.

Angle, Dihedral. In geometry the angle between two planes. The wings of an aeroplane are said to be at a dihedral angle when both right and left wings are upwardly or downwardly inclined to a horizontal transverse line. The angle is measured by the inclination of each wing to the horizontal. If the inclination is upward the angle is said to be positive, if downward, negative.

Angle, Gliding. The angle between the horizontal and the path along which an aeroplane in ordinary flying attitude, but not under engine power, descends in still air.

Angle of Incidence or Angle of Attack. The angle a wing makes with the direction of its motion relative to the air. The angle is usually measured between the chord of the wing and the direction of motion, when parallel to the plane of symmetry of the machine.

Attitude. An aeroplane's or wing's position relative to the direction of motion through the air.

Back, To. Of the wind, to change direction counter-sunwise (counter-clockwise).

Balancing Flaps. Aerofoils used for causing an aeroplane to roll about its longitudinal axis for the purpose of balancing.

Ballonet. A word taken from the French, meaning "a little balloon," and exclusively limited to an interior bag containing air within the envelope of an airship.

Bank, To. To heel for the purpose of turning.

Biplane. Aeroplane with two lifting surfaces, one above the other.

Body. Of an aeroplane—that part which usually contains the engine, crew, tanks, etc., and to which the wings, carriage and other organs are attached.

Bracing. A system of struts and ties to transfer a force from one point to another.

Bulkhead. (*a*) In wing structure, a section through a gap strut parallel to the plane of symmetry.

(*b*) In a fuselage constructed of girders of the N type, a section through the struts perpendicular to the axis of the machine.

Cabane. A French word to denote the mast structure projecting above the body, to which the top load wires of a monoplane are attached.

Cabre. Tail down.

Camber (of a wing section). The convexity of a wing section. The camber is usually measured (as a fraction of the chord) by the maximum height above the chord.

Cant. To tilt; to take any inclined position.

Carriage. That part of the aircraft beneath the body intended for its support, on land or water, and to absorb the shock of alighting.

Ceiling. Maximum height attainable by an aeroplane or airship.

Chassis. See "Carriage."

Chord. The straight line joining the leading and trailing edges of an aerofoil and parallel to the plane of symmetry.

Clinometer. See "Inclinometer."

Control Lever. On an aeroplane, a lever by means of which the principal controls are worked. It usually controls pitching and rolling.

Cross-section. Of an aerofoil. The section cut by a fore and aft plane normal to the surface (commonly the under-surface).

Cross-wind Force. Component of resultant wind force perpendicular to the lift and drag.

Decalage. The difference in angle of incidence between the two planes of a biplane.

Dihedral Angle. See under "Angle."

Dive. To descend steeply with the nose of the aircraft down.

Dope, To. Of fabrics—to paint a fabric with a fluid substance for the purpose of tightening and protecting it.

Drag. The resistance along the line of flight; the head resistance. Compare "Drift."

Drift. The distance drifted. The speed of drifting. The word "drift" having a well-accepted nautical significance should be avoided as far as possible in the sense of head resistance or drag.

Drift, To. To be carried by a current of air or water; to make leeway.

Elevator. An aerofoil set in a more or less horizontal plane and hinged on an athwartships or transverse line. It is used for controlling the angle of incidence of the aeroplane.

Entering Edge. See "Leading Edge."

Fairing. A piece added to any structure to reduce its head resistance or drag.

Fins. Subsidiary aerofoils set parallel to the normal direction of motion of an aircraft.

Flaps, Balancing. See under "Balancing Flaps."

Flaps, Wing. See under "Balancing Flaps."

Fuselage. A covered-in tapering structure connecting the tail system to the remainder of the machine (cf. "Outrigger").

Gap. The distance between the upper and lower wings of a biplane. For specific purposes the points between which it is measured should be indicated.

Glide, To. To fly, usually on a descending path, when the aircraft is not under engine power.

Gliding Angle. See under "Angle."

Incidence, Angle of. See under "Angle."

Inclinometer. An instrument for measuring angle of slope of an aircraft, referred to the horizontal.

Lateral Force. Component of resultant force perpendicular to plane of symmetry.

Leading Edge. Of a wing—the forward edge.

Leeward. Away from the wind.

Leeway. Lateral drift to leeward.

Lift. The force exerted by the air on an aerofoil in a direction perpendicular to the motion, and in the plane of symmetry. Usually upwards in ordinary flight.

Longeron. See "Longitudinals."

Longitudinal Force. Component of resultant force parallel to the axis of the machine.

Longitudinals. Of an aeroplane, the long fore and aft spars connecting the main with the subsidiary supporting or controlling surfaces.

Monoplane. Machine with a single lifting surface.

Normal Force. Component of resultant force perpendicular to axis of machine and parallel to the plane of symmetry.

Outrigger. An open framework used for supporting the tail system.

Pancake, To. To descend steeply with the wings at a very large angle of incidence, like a parachute. Contrast "Dive."

Phugoid. A type of fore and aft oscillation as the result of a disturbance.

Pitch, To. To plunge in the fore and aft direction (nose up or down). Contrast this with "Roll."

Pitot Tube. A tube with open end facing wind which, combined with a static pressure or suction tube, is used in conjunction with a gauge to measure fluid pressures and velocities.

Pressure Tube, Static. A tube (usually with holes in its side past which the fluid flows) so designed that the pressure inside it equals the pressure exerted by the fluid on any body at rest in the fluid. Used as part of a pressure head.

Propeller. An airscrew behind the main supporting surfaces. Compare "Tractor."

Pylon. A mast or post.

Rib. Of a wing, a light fore and aft member which carries the fabric for the purpose of giving the desired cross-section to the wing.

Rib, Compression. A rib designed to act as a strut between the front and rear spars of a wing.

Roll, To. To turn about the fore and aft axis.

Rudder. A subsidiary aerofoil (in an aeroplane more or less perpendicular to the main supporting surfaces) by means of which the aircraft is turned to right or left.

Rudder Bar. The foot-bar, by means of which the rudder of an aeroplane is worked.

Rudder Post. The main post of a rudder.

Screw, Air. An aerofoil so shaped that its rotation about an axis produces a force along that axis for driving an aircraft. (See "Propeller" and "Tractor.")

Side Drift. See "Drift."

Side Slip, To. In an aircraft, to move more or less broadside on relative to the air.

Skid. A part of the alighting gear of an aircraft arranged to slide along the ground under the tail.

Span of Aeroplanes. The maximum transverse dimension.

Span of Wings. The distance from wing tip to wing tip.

Spar. A long piece of timber or other material. In a wing, either of the beams which run transverse to the aircraft, and transfer the lift from the ribs to the frame and bracing.

Stagger. Of wings. When the wings of a biplane are set with the upper one slightly ahead of or abaft of the other, they are said to be staggered. The stagger is measured by the angle made by the line joining the leading edges with the normal to the fore and aft axis of the aeroplane, or sometimes normal to the chord. It is convenient to call the stagger positive if the upper wing is ahead of the lower.

Stalling Angle. Angle of attack of the wings at the minimum speed of flight.

Static Pressure Tube. See under "Pressure."

Statoscope. An instrument to detect the existence of a small rate of ascent or descent.

Strainer. An appliance bearing a suitable mesh for straining impurities from petrol and other fluids.

Stream line. A line traced out in the fluid indicating the direction of motion of the fluid at that instant.

Strut. A structural member intended to resist compression in the direction of its length.

Strut, Gap or Interplane. The vertical struts connecting the wings of a multiplane.

Tail. The after-part of an aircraft, usually carrying certain controlling organs.

Tie. A structural member intended to resist tension.

Top Load Wires. See under "Wires."

Top Surface Camber. See under "Camber."

Top Warp Wires. See under "Wires."

Tractor. An airscrew mounted in front of the main supporting surfaces.

Tractor Machine. An aeroplane with airscrew mounted in front of the main supporting surfaces.

Trailing Edge of a Wing. The after edge.

Triplane. A machine with three lifting surfaces one above the other.

Turnbuckle. A form of wire tightener.

Undercarriage. See "Carriage."

Under Surface Camber. See "Camber."

Veer of the Wind. To change direction sunwise (clockwise).

Velocity of Sideslip. The speed with which the craft moves broadside on with respect to the air. Distinguish from "Drift," q.v.

Warp, To. Of a wing, to bend so that the outer end of the back spar moves up or down. It is convenient to call the warp positive when the movement is downwards.

Wing Flaps. See "Balancing Flaps."

Wings. The main supporting organs of an aeroplane.

Wires, Anti-lift. Wires, the principal function of which is to take the vertical load due to the weight when the machine is not in flight.

Wires, Drag. Wires, the principal function of which is to transfer the drag of the wings to the body or other part of the aeroplane structure. Wires intended mainly to resist forces in the opposite direction to the drag are sometimes called "anti-drag wires."

Wires, Drift. See "Wires, Drag."

Wires, Incidence. The cross-bracing wires between the wings parallel to the plane of symmetry.

Wires, Lift. Wires, the principal function of which is to transfer the lift of the wings to the body or other part of the aeroplane structure.

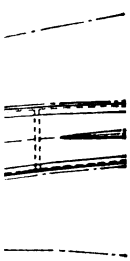
Wires, Top Load. Wires intended mainly to resist forces in the opposite direction to the lift.

Wires, Top Warp. Top load wires connected to the back spar and passing from wing to wing to allow the wings to warp.

Wires, Warp. Lift wires connected to the back spar and controlled so as to move its outer end down for the purpose of warping the wing.

Wire-strainer. See "Turnbuckle."

Yaw, To. An aircraft is said to yaw when its fore and aft axis turns to right or left out of the line of flight. The angle of yaw is the angle between the fore and aft axis of the aircraft and the instantaneous line of flight.



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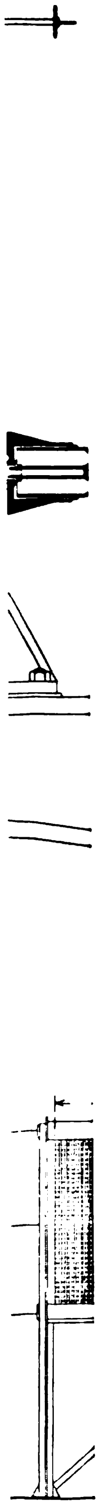

INDEX